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V.N. Branets and I.P. Shmyglevskiy

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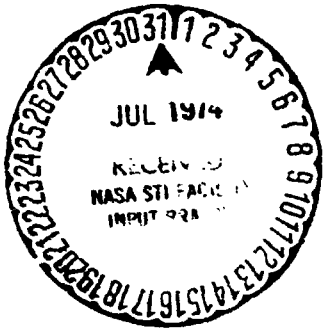

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16. Abstract

The book is devoted to the questions of applying quaternions to general theoretical problems of the kinematics of rigid body rotation and to practical problems of control. The theory of finite rotations and the kinematics of rigid body rotation are presented on the basis of quaternion product operations. Questions in the study of the kinematic equations and their numerical integration, problems in the use of quaternions in problems of rigid body rotation control, and optimization of spatial turns are considered. A number of examples indicate the convenience and advantage of using quaternions in control problems. The book will be of interest to specialists in the area of aircraft control systems and for everyone interested in general problems of rigid body mechanics.

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PREFACE

The basic results of the classical theory of rigid body rotation were obtained in the last century. A number of kinematic parameters, such as the Rodriquez-Hamilton parameters, the Cayley-Klein parameters, the Euler-Krylov angles and the direction cosines, were proposed for describing the motion of a rigid body about a fixed point. At the same time, the attention of the investigators was directed at finding a convenient form for representing the equations and finding integrable cases of motion. After this, the emphasis of the investigations shifted into the area of the applied theory of gyroscopes. The rigid body motion was specified exclusively by the Euler-Krylov angles, which found wide application in all works on mechanics, including works on investigations of the control of moving objects. In addition, the structure of matrices was also applied in theoretical works, the position of the body was specified by the direction cosines. The other methods for describing angular motion were mentioned mainly to assure a comprehensive discussion.

The situation has changed over the last decade. The development of aircraft control systems and the use of computers in motion control have led to the result that the rational description of the spatial motion of the rigid body in various control problems has acquired practical value. The construction of platformless (Cardan-less) inertial systems and the construction of control systems for spatial turns, orientation and stabilization of rigid bodies are related, in particular, to such problems. There are some disadvantages in using the Euler-Krylov angles in these problems. In essence, any system of angles simulates some Cardan suspension. Degeneracy of the kinematic equations occurs for

specific values of the angles, which reflects the effect of folding this suspension frame. Such a degeneracy of the equations is not caused by real physical limitations imposed on the angular motions of the rigid body. In addition, integration of the kinematic equations and transformation of the coordinates in the Euler-Krylov angles are related to trigonometric operations, which reduce the efficiency of using computers.

Among all the kinematic parameters, the Rodriguez-Hamilton parameters and the Cayley-Klein parameters occupy a special place. These parameters are not degenerate for any position of the rigid body, in contrast to the Euler angles. There are four of these parameters. Thus they have one coupling equation, in contrast to the six for the direction cosines. All of this simplifies the problem of integrating numerically the kinematic equations. In addition, the Rodriguez-Hamilton (and the Cayley-Klein) parameters make it possible to efficiently solve many problems in the theory of finite rigid body rotations, stability of controlled motion, etc. In spite of such advantages, the use of these parameters is complicated by the fact that it leads to the application of the vector theory of finite rotations or even to stereographic projection and unitary transformations, which do not have a simple geometric meaning. The use of quaternions permits the creation of a very convenient and clear formalism which uses the Rodriguez-Hamilton parameters.

Quaternions were first introduced in mathematics by W.R. Hamilton in 1843. Starting in 1833, he studied the theory of the quaternions created by him, trying to find for the study of spatial geometry the same sort of convenient structure as complex numbers used in solving planar problems. The results of his work appeared in two books: "Lectures on Quaternions" [55] and "Elements of Quaternions" [54]. Subsequently, the theory of quaternions was studied by H. Hankel and F. Klein. However, until recently qua-

ternions had not found any practical application and had served as an example of a formal mathematical model of four-dimensional linear algebra.

Works of the authors on applying quaternions to the study of rigid body rotation are presented in this monograph. The formalism of orthogonal transformations and theory of finite rotations of rigid bodies is developed on the basis of the known form of representing orthogonal transformations in the form of quaternion product operations. An attempt is made to relate the developed formalism using quaternions to other kinematic parameters. A study of the general properties of the kinematic equations is carried out in detail. A general method for deriving these equations is proposed, which is valid for the various kinematic parameters and well illustrates their physical meaning.

The application of quaternions permits one to represent in a unique vector form infinitesimal rotations, which define the angular velocity vector, and arbitrary transformations, which are finite rotations. Quaternions have an extremely convenient structure for studying the kinematics of rigid body motion, which is explained by the dualism of the quaternion units, which are, on the one hand, bases of real three-dimensional space and, on the other, transformation operators. Due to this property of quaternions, the Rodriguez-Hamilton and the Cayley-Klein parameters have a simple and clear physical meaning.

Questions of applying the kinematics of angular motion of practical problems of rigid body motion control are also considered in the book. Questions of integrating numerically the kinematic equations and questions of using the kinematic parameters (in particular, the quaternion elements) in control are studied in detail, and particular cases of problems in optimizing spatial turns are also considered. The authors have tried to indicate the

possibilities of applying the quaterion structure both to general theoretical problems and to practical problems of rigid body motion control, and also to draw to the attention of a wide circle of specialists working in the area of aircraft control systems and platformless systems, and everyone interested in general problems of rigid body mechanics, to this powerful structure.

The authors take this opportunity to express their deep appreciation to B.V. Raushenbakh and T.M. Eneyev for help i work on this book.

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APPLICATION OF QUATERNIONS TO RIGID BODY ROTATION PROBLEMS

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CHAPTER 1

QUATERNIONS AND THEIR PROPERTIES

/11*

§ 1.1. QUANTERNION ALGEBRA

The necessity of extending the operations of three-dimensional vector algebra to the operations of multiplication and division led Hamilton (1843) to introduce an algebra for four-dimensional numbers, or quaternions [54, 55]. Quaternion means a number consisting of a real unit 1 and three imaginary units ** i_1, i_2, i_3, with real elements of the following form:

$$\Lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \lambda_0 1 + \lambda_1 i_1 + \lambda_2 i_2 + \lambda_3 i_3.$$

We will state the basic postulates defining the quaternion operations.

1° Two quaternions Λ and M are equal if their elements $\lambda_i = \mu_i$ $(i = 0, 1, 2, 3)$ are equal.

2° The quaternion whose elements are the quantities $\lambda_i + \mu_i$ is called the sum of the quaternions Λ and M:

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* Numbers in margin indicate pagination in original foreign text.

** Hamilton denoted the three imaginary units by i, j, k. However, for convenience, we will use the notation introduced by Hankel: i_1, i_2, i_3. We note that it is possible to extend the hypercomplex numbers further to the case when the λ_i themselves are also complex. The numbers constructed in such a manner are called biquaternions and have found application for describing helical motion of rigid bodies. Biquaternions will not be considered in this work.

$$\Lambda + \mathrm{M} = (\lambda_0 + \mu_0)\,1 + (\lambda_1 + \mu_1)\,i_1 + (\lambda_2 + \mu_2)\,i_2 + (\lambda_3 + \mu_3)\,i_3.$$

3° Multiplication of the quaternion Λ by the scalar a is performed by multiplying all its elements by this number:

$$a\Lambda = a\lambda_0 1 + a\lambda_1 i_1 + a\lambda_2 i_2 + a\lambda_3 i_3.$$

In particular, the quaternion:

$$-\Lambda = -\lambda_0 1 - \lambda_1 i_1 - \lambda_2 i_2 - \lambda_3 i_3,$$

will be negative, and the quaternion (0, 0, 0, 0) will be zero.

It follows from these definitions that the addition of quaterions and their multiplication by a scalar satisfy the rules of ordinary algebra: /12

4° $\Lambda + \mathrm{M} = \mathrm{M} + \Lambda$, $(\Lambda + \mathrm{M}) + \mathrm{N} = \Lambda + (\mathrm{M} + \mathrm{N})$;

5° $a\Lambda = \Lambda a$, $(ab)\Lambda = \Lambda(ba)$;

6° $(a + b)\Lambda = a\Lambda + b\Lambda$, $a(\Lambda + \mathrm{M}) = a\Lambda + a\mathrm{M}$.

The units 1, i_1, i_2, i_3 can be considered as unit vectors (basis vectors) of a four-dimensional space, which we denote by H. Then any quaternion can be represented in this space by a point or a radius vector. Addition of vectors and their multiplication by a scalar in the space H are performed in the same way as in ordinary vector space. The peculiarity of the space H consists of the fact that it is closed relative to the operations of multiplication and division.

7° In order to define the product of quaternions, it is necessary to specify the rules for multiplying the units 1, ary to i_1, i_2, i_3. These rules are the following:

$$1 \circ i_1 = i_1 \circ 1 = i_1,\ 1 \circ i_2 = i_2 \circ 1 = i_2,\ 1 \circ i_3 = i_3 \circ 1 = i_3,\ 1 \circ 1 = 1,$$
$$i_1 \circ i_1 = -1, \quad i_2 \circ i_2 = -1, \quad i_3 \circ i_3 = -1,$$
$$i_1 \circ i_2 = -i_2 \circ i_1 = i_3,\ i_3 \circ i_1 = -i_1 \circ i_3 = i_2,\ i_2 \circ i_3 = -i_3 \circ i_2 = i_1.$$

With such a rule for multiplication, the product of two quaternions is also a quaternion.

The rules for multiplication are extremely fortunate *; thanks to them, quaternion algebra includes the algebra of real and complex numbers, and also three-dimensional vector algebra.

Quaternions contain the real numbers (a, 0, 0, 0) with the single unit 1, the complex numbers (a, b, 0, 0) with the two units 1, i, and the vectors (0, a, b, c) in three-dimensional space. Although the real and complex numbers form a field (i.e., addition, multiplication and division give back elements of the same set), the product of two vectors, as will be shown below, is already not a vector, but a quaternion. /13

The rule for multiplication 7° indicates that multiplication by 1 does not change a quaternion, i.e., the component 1 behaves as an ordinary scalar. For this reason, we will subsequently write the first term (λ_0) without the unit in expressions for quaternions.

One can further identify the units i_1, i_2, i_3 with the unit vectors of three-dimensional vector space ** and consider the coefficients with these units as vector components. In accordance

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* Hamilton wrote about the discovery of quaternions: "They were born completely formulated on October 16, 1843, when Mrs. Hamilton and I were strolling about Dublin and were climbing Brougham bridge, which students have called Quaternion bridge since then. My chain of thought was closed at this moment and the basic relations of i, j, k arose vividly exactly as I use them now."

** Since, if the orthogonal transformation of i_i into i'_j ($i'_j = c_{ji} i_i$) is introduced, then because of the orthogonality $c_{ji} c_{ki} = \delta_{jk}$ it can be seen that i_j satisfies the same multiplication rules as the quaternion units:

$$i'_1 \circ i'_1 = -c_{1i} c_{1i} = -1, \quad i'_2 \circ i'_3 = -c_{2i} c_{3i} + \begin{vmatrix} i_1 & i_2 & i_3 \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} =$$

$$= c_{11} i_1 + c_{12} i_2 + c_{13} i_3 = i'_1 \text{ etc.}$$

with this we will represent a quaternion in the form of a sum of scalar and vector parts, which will be denoted by sqal Λ and vect Λ respectively:

8° $\Lambda = \text{sqal}\,\Lambda + \text{vect}\,\Lambda = \lambda_0 + \boldsymbol{\lambda}$ The expression for the vector part will be written in the form taken for ordinary vectors for convenience.

The rules for multiplication of the imaginary units are easily remembered by using the following representation (Figure 1.1): when multiplying two units arranged clockwise, one obtains the third unit with a plus sign; when moving in the opposite direction counterclockwise) the unit is taken with a minus sign.

By performing the multiplication of two quaternions Λ and $\mathbf{M}$, /14 we obtain:

$$\Lambda \circ \mathbf{M} = \lambda_0\mu_0 - \lambda_1\mu_1 - \lambda_2\mu_2 - \lambda_3\mu_3 + \lambda_0(\mu_1 i_1 + \mu_2 i_2 + \mu_3 i_3) + \mu_0(\lambda_1 i_1 + \lambda_2 i_2 + \lambda_3 i_3) + \begin{vmatrix} i_1 & i_2 & i_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix}. \quad (1.1)$$

If the order of the factors is changed, i.e. the product $\mathbf{M} \circ \Lambda$ is considered, then the determinant in the multiplication formula (1.1) changes; the lines of λ and μ are interchanged (i.e., the determinant changes sign). By using formula (1.1), one can prove the basic properties of multiplication.

9° Quaternion multiplication is associative and distributive with respect to addition.

For the proof of this statement, we consider the product of the three imaginary units, since this property is evident for the real unit. We have:

$$(i_1 \circ i_2) \circ i_3 = i_3 \circ i_3 = -1, \quad i_1 \circ (i_2 \circ i_3) = i_1 \circ i_1 = -1,$$
$$(i_1 \circ i_3) \circ i_2 = -i_2 \circ i_2 = 1, \quad i_1 \circ (i_3 \circ i_2) = -i_1 \circ i_1 = 1,$$
$$(i_1 \circ i_2) \circ i_1 = i_3 \circ i_1 = i_2, \quad i_1 \circ (i_2 \circ i_1) = -i_1 \circ i_3 = i_2,$$
$$(i_1 \circ i_1) \circ i_2 = -i_2, \quad i_1 \circ (i_1 \circ i_2) = i_1 \circ i_3 = -i_2,$$
$$(i_1 \circ i_3) \circ i_1 = -i_2 \circ i_1 = i_3, \quad i_1 \circ (i_3 \circ i_1) = i_1 \circ i_2 = i_3,$$
$$(i_1 \circ i_1) \circ i_3 = -i_3, \quad i_1 \circ (i_1 \circ i_3) = -i_1 \circ i_2 = -i_3;$$

the remaining products are obtained by analogy by cyclic permutation (the possibility of which follows from the multiplication rules 7° illustrated by Figure 1.1). Hence, it follows that the equality:

$$(\Lambda \circ M) \circ N = \Lambda \circ (M \circ N)$$

is correct for any quaternions. It can be verified in precisely the same manner that:

$$\Lambda \circ (M + N) = \Lambda \circ M + \Lambda \circ N,$$

since the components of the products on both sides of the equality coincide for each unit.

10° Quaternion multiplication is noncommutative; permutation of the factors is possible only when one of the factors is a scalar or when the vector parts of the factors are proportional.

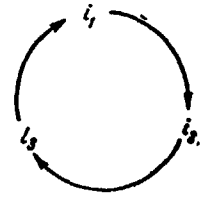

Figure 1.1.

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Actually, by examining the multiplication formula (1.1) one can see that $\Lambda \circ M = M \circ \Lambda$ only when the determinant vanishes. This is possible either when $\lambda_1 = \lambda_2 = \lambda_3 = 0$ or $\mu_1 = \mu_2 = \mu_3 = 0$, i.e. when Λ or M is a scalar, or when $\frac{\lambda_1}{\mu_1} = \frac{\lambda_2}{\mu_2} = \frac{\lambda_3}{\mu_3} = a$, i.e. $\lambda = a\mu$.

We also have from formula (1.1):

$$\text{sqal}(\Lambda \circ M) = \text{sqal}(M \circ \Lambda).$$

By using this relation, one can show that:

11° The scalar part of a quaternion product does not change with cyclic permutation of the factors.

By considering the equality:

$$\text{sqal}[\Lambda \circ (M \circ N)] = \text{sqal}[(M \circ N) \circ \Lambda] = \cdot = \text{sqal}[M \circ (N \circ \Lambda)] = \text{sqal}[(N \circ \Lambda) \circ M],$$

we obtain that:

$$\text{sqal}[\Lambda \circ M \circ N] = \text{sqal}[M \circ N \circ \Lambda] = \text{sqal}[N \circ \Lambda \circ M]. \qquad (1.2)$$

12° The following quaternion denoted by $\tilde{\Lambda}$ is the conjugate of the given quaternion Λ:

$$\tilde{\Lambda} = \lambda_0 - \lambda.$$

One can see that the conjugate of the sum of quaternions equals the sum of the conjugates, i.e.:

$$\widetilde{(\Lambda + M)} = \tilde{\Lambda} + \tilde{M}.$$

Since the vector parts of the quaternions Λ and $\tilde{\Lambda}$ differ in sign, it follows from property 10° that:

$$\Lambda \circ \tilde{\Lambda} = \tilde{\Lambda} \circ \Lambda.$$

This product is called the norm of the quaternion Λ and is denoted by $\|\Lambda\|$. It follows from the multiplication formula (1.1) that:

13° $$\|\Lambda\| = \Lambda \circ \tilde{\Lambda} = \tilde{\Lambda} \circ \Lambda = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2.$$

Thus, the norm of a quaternion is a scalar; $\|\Lambda\| = 0$ if and only if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_0 = 0$ *. When the norm $\|\Lambda\| = 1$, the quaternion is called normalized. /16

By examining the product (1.1), one can see that changing the signs of the vector parts of the factors changes the sign for the terms with λ_0 and μ_0 and does not change the sign of the determinant. By interchanging the rows of λ and μ, we obtain the quaternion whose vector part has changed sign: $\widetilde{(\Lambda \circ M)}$. It is evident that it is

* This is not satisfied for biquaternions.

obtained by multiplying $\tilde{M}$ by $\tilde{\Lambda}$:

14° $\overline{(\Lambda \circ M)} = \tilde{M} \circ \tilde{\Lambda}$, , i.e., the conjugate of the product of two quaternions equals the product of their conjugates taken in the reverse order.

By using the obtained rule, we find the value of the norm of a product:

15° $\|\Lambda \circ M\| = \Lambda \circ M \circ \overline{(\Lambda \circ M)} = \Lambda \circ M \circ \tilde{M} \circ \tilde{\Lambda} = \Lambda \circ \|M\| \circ \tilde{\Lambda} = \|M\| \circ \Lambda \circ \tilde{\Lambda} = \|M\| \circ \|\Lambda\|$ since $\|M\|$ is a scalar. Hence, it follows that the norm of the product of two quaternions equals the product of the norms of the factors.

It can be shown by induction that the properties 14° and 15° are preserved for n factors:

$$\overline{(\Lambda_1 \circ \Lambda_2 \circ \ldots \circ \Lambda_n)} = \tilde{\Lambda}_n \circ \tilde{\Lambda}_{n-1} \circ \ldots \circ \tilde{\Lambda}_1, \quad (1.3)$$

$$\|\Lambda_1 \circ \Lambda_2 \circ \ldots \circ \Lambda_n\| = \|\Lambda_1\| \|\Lambda_2\| \ldots \|\Lambda_n\|. \quad (1.4)$$

The following can also be obtained from the obtained properties:

16° The product of two quaternions equals zero only if one of the factors equals zero.

Actually, if $\Lambda \circ M = 0$, then $\|\Lambda \circ M\| = 0$, hence $\|\Lambda\| \|M\| = 0$. Since $\|\Lambda\|$ and $\|M\|$ are scalars, then the last relation indicates that either $\|\Lambda\| = 0$ or $\|M\| = 0$, which is possible if $\Lambda = 0$ or $M = 0$ respectively. It follows from this same property that the product /17 of normalized quaternions is also a normalized quaternion.

Let two known quaternions Λ and M be given and the quaternion N be defined as the result of their multiplication, i.e.:

$$N = \Lambda \circ M.$$

By equating the elements for the units i_1, i_2, i_3 in the multiplication formula (1.1), we have:

$$\left.\begin{aligned} v_0 &= \lambda_0\mu_0 - \lambda_1\mu_1 - \lambda_2\mu_2 - \lambda_3\mu_3, \\ v_1 &= \lambda_0\mu_1 + \lambda_1\mu_0 + \lambda_2\mu_3 - \lambda_3\mu_2, \\ v_2 &= \lambda_0\mu_2 + \lambda_2\mu_0 + \lambda_3\mu_1 - \lambda_1\mu_3, \\ v_3 &= \lambda_0\mu_3 + \lambda_3\mu_0 + \lambda_1\mu_2 - \lambda_2\mu_1. \end{aligned}\right\} \qquad (1.5)$$

If the quaternions **N** and **A** (or **M**) are specified, then to determine the quaternion **M** (or **A**) it is necessary to define the operation of quaternion division.

In the coordinates this operation is equivalent to solving the system of linear algebraic equations (1.5) for the unknown u_i (or λ_i); solution of the system is possible if the determinant of the system does not equal zero.

If **A** is a quaternion not equaling zero, i.e. $|A| \neq 0$, then following the definition of the norm 13°, we obtain:

$$A \circ \frac{\bar{A}}{|A|} = \frac{\bar{A}}{|A|} \circ A = 1.$$

17° The quaternion $A^{-1} = \frac{\bar{A}}{|A|}$ for which the equality:

$$A \circ A^{-1} = A^{-1} \circ A = 1.$$

is satisfied is called the inverse of the given quaternion **A**.

Hence, the equation $N = A \circ M$ can be solved for the unknown **M** in the following manner: we multiply the given equality from the left by A^{-1}, as a result of which we obtain:

$$A^{-1} \circ A \circ M = M = A^{-1} \circ N = \frac{\bar{A} \circ N}{|A|}.$$

For **A** unknown, we obtain by analogy:

$$A = N \circ M^{-} = \frac{N \circ \bar{M}}{|M|}.$$

The solution found corresponds to the solution of the system /18 of linear equations (1.5).

It should be noted that the formulae for obtaining Λ and $\mathbf{M}$ are not symmetric; this is related to the noncommutativity of multiplication. For this reason it is impossible to use a symmetric notation for quaternion division in the form $\frac{\mathbf{N}}{\Lambda}$ or $\frac{\mathbf{N}}{\mathbf{M}}$. We note also that the obtained solutions are unique, because if there were another solution, for example $\mathbf{M}_1 = \Lambda^{-1} \circ \mathbf{N}$, then the relation $\Lambda \circ \mathbf{M}_1 = \Lambda \circ \mathbf{M}$ would be true, i.e., $\Lambda \circ (\mathbf{M}_1 - \mathbf{M}) = 0$. Hence, it follows that $\mathbf{M}_1 - \mathbf{M} = 0$ (cf. 16°).

Property 15° indicates that if $\mathbf{M} = \Lambda^{-1} \circ \mathbf{N}$, then their norms satisfy the equality $\|\mathbf{M}\| = \frac{\|\mathbf{N}\|}{\|\Lambda\|}$.

The inverse of a quaternion product is found in the following manner:

$$\mathbf{N}^{-1} = \frac{\tilde{\mathbf{N}}}{\|\mathbf{N}\|} = \frac{\widetilde{(\Lambda \circ \mathbf{M})}}{\|\Lambda \circ \mathbf{M}\|} = \frac{\tilde{\mathbf{M}} \circ \tilde{\Lambda}}{\|\mathbf{M}\| \cdot \|\Lambda\|} = \mathbf{M}^{-1} \circ \Lambda^{-1}$$

and:

$$(\Lambda_1 \circ \Lambda_2 \circ \ldots \circ \Lambda_n)^{-1} = \frac{\widetilde{(\Lambda_1 \circ \Lambda_2 \circ \ldots \circ \Lambda_n)}}{\|\Lambda_1 \circ \Lambda_2 \circ \ldots \circ \Lambda_n\|} = \Lambda_n^{-1} \circ \Lambda_{n-1}^{-1} \circ \ldots \circ \Lambda_1^{-1}. \tag{1.6}$$

We now consider the product of quaternions having zero scalar parts, i.e. the product of "vectors". In accordance with the multiplication formula (1.1), we obtain:

$$\lambda \circ \mu = -\lambda_1\mu_1 - \lambda_2\mu_2 - \lambda_3\mu_3 + \begin{vmatrix} i_1 & i_2 & i_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix}.$$

The scalar and vector parts of the obtained product equal:

$$\operatorname{sqal}(\lambda \circ \mu) = -(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3)$$

and:

$$\text{vect}(\lambda \circ \mu) = \begin{vmatrix} i_1 & i_2 & i_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix}.$$

In order to have a geometric interpretation in the three-dimensional basis i_1, i_2, i_3 (Figure 1.2), we set: /19

$$\lambda = \lambda i_1, \quad \mu = \mu (i_1 \cos\vartheta + i_2 \sin\vartheta).$$

The product of these quaternions equals:

$$\lambda \circ \mu = \lambda\mu (-\cos\vartheta + i_3 \sin\vartheta),$$

hence, we have:

$$\text{sqal}(\lambda \circ \mu) = -\lambda\mu \cos\vartheta,$$
$$\text{vect}(\lambda \circ \mu) = i_3 \lambda\mu \sin\vartheta.$$

By examining the right-handed orthogonal system of axes i_1, i_2, i_3 (cf. Figure 1.2), one can see that the scalar and vector products of two vectors as understood in ordinary vector algebra are equal respectively to:

$$\left.\begin{aligned} \lambda \cdot \mu &= -\text{sqal}(\lambda \circ \mu), \\ \lambda \times \mu &= \text{vect}(\lambda \circ \mu) \end{aligned}\right\} \qquad (1.7)$$

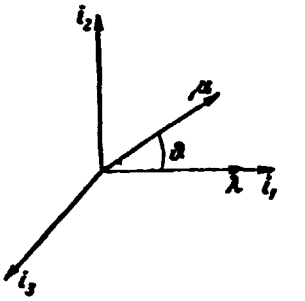

Figure 1.2.

(the point will indicate the scalar product, the sign $\times$ - the vector product, and the sign $\circ$ - quaternion multiplication).

Interchange of the factors does not change the scalar part of the product, but changes the direction of the basis vector i_3. Hence, it follows that:

$$\text{sqal}(\lambda \circ \mu) = \text{sqal}(\mu \circ \lambda), \quad \text{vect}(\lambda \circ \mu) = -\text{vect}(\mu \circ \lambda).$$

One can obtain from these relations:

$$\left.\begin{aligned} 2\,\mathrm{sqal}(\lambda\circ\mu) &= -2\lambda\cdot\mu=\lambda\circ\mu+\mu\circ\lambda,\\ 2\,\mathrm{vect}(\lambda\circ\mu) &= 2\lambda\times\mu=\lambda\circ\mu-\mu\circ\lambda. \end{aligned}\right\} \tag{1.8}$$

By using the symbols for the scalar and vector products, we present the formula for multiplying two quaternions (1.1) in the following form:

$$\left.\begin{aligned} \lambda\circ\mu &= -\lambda\cdot\mu+\lambda\times\mu,\\ \Lambda\circ M &= \lambda_0\mu_0-\lambda\cdot\mu+\lambda_0\mu+\mu_0\lambda+\lambda\times\mu. \end{aligned}\right\} \tag{1.9}$$

These relations establish the correspondence between vector algebra /20 and quaternion algebra.

In conclusion, we examine the formula for multiplying three vectors.

Turning to (1.2), we obtain:

$$\mathrm{sqal}(\lambda\circ\mu\circ\nu)=\mathrm{sqal}(\mu\circ\nu\circ\lambda)=\mathrm{sqal}(\nu\circ\lambda\circ\mu).$$

Since $\overline{(\lambda\circ\mu\circ\nu)}=(-\nu)\circ(-\mu)\circ(-\lambda)=-\nu\circ\mu\circ\lambda$ and the scalar parts of a quaternion and its conjugate coincide, and their vector parts differ in sign, the following equalities are true:

$$\mathrm{sqal}(\overline{\lambda\circ\mu\circ\nu})=\mathrm{sqal}(\lambda\circ\mu\circ\nu)=-\mathrm{sqal}(\nu\circ\mu\circ\lambda),$$

$$\mathrm{vect}(\overline{\lambda\circ\mu\circ\nu})=-\mathrm{vect}(\lambda\circ\mu\circ\nu)=\mathrm{vect}(\nu\circ\mu\circ\lambda).$$

§ 1.2. SPHERICAL REPRESENTATION OF QUATERNIONS

Any quaternion with real elements:

$$\Lambda=\lambda_0+\lambda_1 i_1+\lambda_2 i_2+\lambda_3 i_3$$

can be represented in the following form:

$$\Lambda=\Lambda\left(\frac{\lambda_0}{\Lambda}+\frac{\lambda_1 i_1+\lambda_2 i_2+\lambda_3 i_3}{\Lambda}\right)=\Lambda\left(\frac{\lambda_0}{\Lambda}+\frac{\lambda}{\Lambda}\right). \tag{1.10}$$

where $\Lambda=\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}=\sqrt{\|\Lambda\|}=|\Lambda|$ is called the tensor of the given quaternion. It is evident that the tensor of the unit quaternion equals one. The quantity $\frac{\lambda_0}{\Lambda}+\frac{\lambda}{\Lambda}$ is called the versor of the quaternion; the meaning of the versor will be explained later. For now, we note only that it is a quaternion whose norm equals

one. We introduce the unit vector ζ directed along the vector λ:

$$\zeta=\frac{\lambda}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}=\frac{\lambda_1 i_1+\lambda_2 i_2+\lambda_3 i_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}.$$

Then the vector part of the versor can be written in the form:

$$\frac{\lambda}{\Lambda}=\zeta\frac{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}{\Lambda}.$$

Noting that the square of the scalar part of the versor and /21
the square of the coefficient of ζ sum to one, we introduce the following variables:

$$\cos\theta=\frac{\lambda_0}{\Lambda},\quad \sin\theta=\frac{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}{\Lambda},\quad 0\leqslant\theta<\pi.$$

With this notation the quaternion (1.10) can be represented in the following form:

$$\Lambda=\Lambda(\cos\theta+\zeta\sin\theta). \tag{1.11}$$

In the expression for the quaternion (1.11) the sign of the angle θ is determined by the selection of the sign for the unit vector ζ, i.e., the choice of its direction (in fact, the sign $\pm$ should have been placed in the expressions for ζ and $\sin\theta$); for now, we will not establish a unique choice for the positive direction in measuring the angle θ and for the positive direction of the unit vector ζ.

Representing the quaternion in the form (1.11) is convenient in that the roots of the equation $\Lambda^n=M$ can then be found easily. Actually, since $\zeta\circ\zeta=\zeta^2=-1$ (which can be verified by direct calculation):

$$\Lambda^2=\Lambda\circ\Lambda=\Lambda^2(\cos\theta+\zeta\sin\theta)^2=\Lambda^2(\cos 2\theta+\zeta\sin 2\theta)$$

and the de Moivre formula is valid for any power n:

$$\Lambda^n=\Lambda^n(\cos n\theta+\zeta\sin n\theta).$$

Solution of the equation $\Lambda^n=M$ is possible when the vector μ is directed along the vector ζ; in this case the solution itself is analogous to the solution of the similar equation in the theory of complex variables, and we will not discuss it here.

Representing the quaternion in the form (1.11) permits one to obtain a clear geometric interpretation of the quaternion.

We consider the normalized quaternion:

$$\Lambda = \cos\theta + \zeta \sin\theta.$$

This quaternion can be represented in the form of a product of a certain two vectors $b \circ a^{-1}$ satisfying the following conditions:

1) $a=b$, $a=\sqrt{\|a\|}$, $b=\sqrt{\|b\|}$;

2) the angle between the vectors **a** and **b** equals θ;

3) the plane of **ab** is perpendicular to the unit vector ζ; /22

4) the vectors **a**, **b**, ζ form a right-handed system of axes (Figure 1.3).

Keeping property 1) in mind, we can write:

$$\Lambda = \cos\theta + \zeta \sin\theta = \frac{ab\cos\theta + \zeta ab \sin\theta}{|a|}.$$

The quantities ab cos θ and ζab sin θ can be represented in the form of the scalar and vector products **a·b** and **aχb** respectively (the resultant of the vector product in the right-handed coordinate system is directed along the vector ζ).

Hence, we obtain:

$$\Lambda = \frac{a\cdot b + a\times b}{|a|} = \frac{-\operatorname{sqal}(a\cdot b) + \operatorname{vect}(a\cdot b)}{|a|} =$$
$$= -\frac{\operatorname{sqal}(b\cdot a) + \operatorname{vect}(b\cdot a)}{|a|} = -\frac{b\circ a}{|a|} = \frac{b\circ\tilde{a}}{|a|} = b\circ a^{-1}. \qquad (1.12)$$

It follows from Figure 1.3 that when the vectors **a**, **b**, ζ form a right-handed coordinate system, the angle $\vartheta < \pi$ is measured from **a** to **b** counterclockwise as viewed from the positive ζ axis. The definiteness taken here for the choice of the positive direction of the unit vector ζ and the corresponding positive direction for measuring the angle ϑ is specified only by the rules for the vector multiplication of the vectors **a** and **b**; in principle, one could take the opposite direction for measuring ϑ; etc. However, the convention established here will be followed henceforth.

It follows from relation (1.12) that to each normalized quaternion Λ one can set in correspondence an arc of a great circle on a sphere of radius a=b, which has the direction of the positive angle ϑ and joins the ends of the vectors **a** and **b** (cf. Figure 1.3). Thus, the quaternion Λ can be uniquely represented by the arc of a great circle arc Λ, whose plane is determined by the vector ζ, /23 and length — by the angle ϑ. The direction of the quaternion is specified by the direction of the vector ζ; the position of the arc Λ on its great circle is arbitrary, i.e. the arc is sliding (with an arbitrary origin).

We consider some particular cases.

When $\vartheta = 0$ or $\vartheta = \pi$, the quaternion Λ=1 or Λ=-1 respectively, and the vector ζ can have an arbitrary direction. This case is the exception, since Λ=1 corresponds to any point on the sphere and Λ=-1 — to any half circle. The unit vector ζ corresponds to one-quarter of a great circle. If arc $\Lambda = \mathcal{AB}$ corresponds to the quaternion Λ, then the inverse, i.e. the conjugate, quaternion $\Lambda^{-1} = \bar{\Lambda} = \cos\vartheta - \zeta\sin\vartheta$ corresponds to arc $\bar{\Lambda}$ equaling the arc $\mathcal{BA}$ (i.e. the arc in the opposite direction, because such a direction of the arc in this case is positive with respect to the vector ζ). The quaternion $-\Lambda$ corresponds to the quaternion:

$$-\cos\vartheta - \zeta\sin\vartheta = \cos(\pi - \vartheta) - \zeta\sin(\pi - \vartheta),$$

i.e. the arc $\mathscr{A}'\mathscr{B}$ (cf. Figure 1.3) running from the diametrically opposite point $\mathscr{A}'$ to the point $\mathscr{B}$.

We now consider the operation of quaternion multiplication and find its representation on the sphere.

Let the unit quaternions Λ and M be given. Both of these quaternions can be represented in the form of (1.11) and in the form of arcs of great circles of appropriate lengths. The arc of the quaternion arc Λ is placed on the great circle perpendicular to the vector λ; the arc arc M — on the great circle defined by the vector μ. We set the end of the arc arc Λ and the origin of the arc arc M at the point of intersection $\mathscr{A}$ of the circles (Figure 1.4).

By introducing the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, we obtain: /24

$$\Lambda = b \circ a^{-1}, \quad \mathrm{M} = a \circ c^{-1}.$$

The arc of the great circle joining the points $\mathscr{C}$ and $\mathscr{B}$, defines some quaternion $\mathrm{N} = b \circ c^{-1}$. It is evident that:

$$\mathrm{N} = b \circ c^{-1} = b \circ a^{-1} \circ a \circ c^{-1} = \Lambda \circ \mathrm{M},$$

i.e. the quaternion product corresponds to the geometric sum of the arcs on the sphere: arc M + arc Λ = arc$(\Lambda \circ \mathrm{M})$.

Since the quaternion Λ is a sliding arc on the sphere, Λ also equals $a \circ f^{-1}$ (the angles $\mathscr{FOA}$ and $\mathscr{AOB}$ are equal and are determined by the angle ϑ); by analogy the quaternion M corresponds to $d \circ a^{-1}$. Hence, it follows that the product $\mathrm{M} \circ \Lambda = d \circ a^{-1} \circ a \circ f^{-1} = d \circ f^{-1}$ corresponds to the arc $\mathscr{FD}$ representing the quaternion $\mathrm{N}' = \mathrm{M} \circ \Lambda$, i.e.:

$$\mathrm{arc}\,\Lambda + \mathrm{arc}\,\mathrm{M} = \mathrm{arc}\,(\mathrm{M} \circ \Lambda).$$

It is particularly clearly seen here what happens when the factors are permuted in quaternion multiplication.

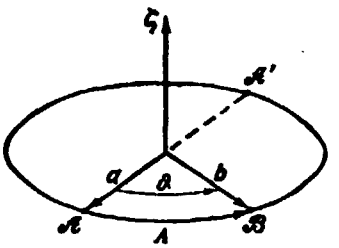

Figure 1.3.

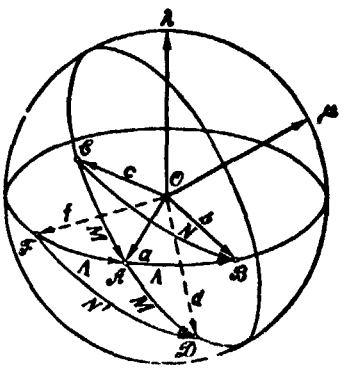

Figure 1.4.

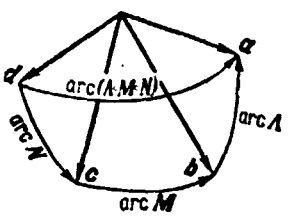

Figure 1.5.

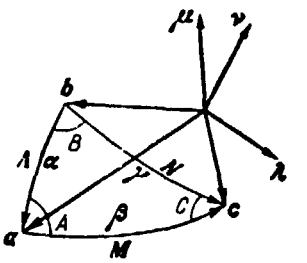

Figure 1.6.

By analogy we have for the geometric sum of three arcs of great circles Λ, M and N (Figure 1.5):

$$\operatorname{arc} N + \operatorname{arc} M + \operatorname{arc} \Lambda = \operatorname{arc}(\Lambda \circ M \circ N),$$

since:

$$\Lambda = a \circ b^{-1}, \quad M = b \circ c^{-1},$$
$$N = c \circ d^{-1}, \quad \Lambda \circ M \circ N = a \circ d^{-1}.$$

Thus, the vector sum of several arcs of great circles, each of which is specified by a quaternion, gives the arc of the great circle which is determined by the product of these quaternions in reverse order.

It follows from the condition that a point on the sphere corresponds to a quaternion equaling zero that for a closed spherical triangle: /25

$$\operatorname{arc} N + \operatorname{arc} M + \operatorname{arc} \Lambda = 0,$$

i.e.

$$\arc \Lambda \circ \mathbf{M} \circ \mathbf{N} = 0 \quad \text{and} \quad \Lambda \circ \mathbf{M} \circ \mathbf{N} = 1.$$

In general, arcs, which represent the cyclically arranged quaternions $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$, form a closed polygon if and only if:

$$\Lambda_n \circ \Lambda_{n-1} \circ \ldots \circ \Lambda_1 = 1.$$

The representation of quaternions in the form of arcs on a sphere presented in this paragraph is very useful in many cases. As an example, we derive the trigonometric formulae of a spherical triangle. Let the following be given for the triangle (Figure 1.6):

$$\left.\begin{aligned} &\arc \Lambda + \arc \mathbf{M} = \arc \mathbf{N} = \arc \mathbf{M} \circ \Lambda, \\ &\Lambda = a \circ b^{-1} = \cos\alpha + \lambda \sin\alpha, \\ &\mathbf{M} = c \circ a^{-1} = \cos\beta + \mu \sin\beta, \\ &\mathbf{N} = c \circ b^{-1} = \cos\gamma + \nu \sin\gamma, \end{aligned}\right\} \qquad (1.13)$$

where α, β, γ are the arc lengths of the quaternions Λ, $\mathbf{M}$ and $\mathbf{N}$.

Since $\mathbf{N} = \mathbf{M} \circ \Lambda$, i.e. $c \circ b^{-1} = (c \circ a^{-1}) \circ (a \circ b^{-1})$, then:

$$(\cos\beta + \mu \sin\beta) \circ (\cos\alpha + \lambda \sin\alpha) = \cos\gamma + \nu \sin\gamma.$$

We note that:

$$\mu \circ \lambda = -\mu \cdot \lambda + \mu \times \lambda = -\lambda \cdot \mu - \lambda \times \mu = \cos\measuredangle - a \sin\measuredangle,$$

where $(\pi - \measuredangle)$ is the angle between the vectors μ and λ, i.e., the angle at the apex a of the triangle equals $\measuredangle$ (cf. Figure 1.6).

By performing the multiplication in (1.13) and equating the scalar parts, we obtain at once the well-known law of cosines:

$$\cos\gamma = \cos\beta \cos\alpha + \sin\beta \sin\alpha \cos\measuredangle.$$

By equating the vector parts of the product, we obtain: /26

$$\mu \sin\beta \cos\alpha + \lambda \sin\alpha \cos\beta - a \sin\measuredangle \sin\beta \sin\alpha = \nu \sin\gamma.$$

We take the scalar product of this equality with $\boldsymbol{a}$; since $\boldsymbol{a}\cdot\boldsymbol{\mu}=\boldsymbol{a}\cdot\boldsymbol{\lambda}=0$ and $\boldsymbol{a}\cdot\boldsymbol{a}=1$, we obtain:

$$\sin\mathscr{A}\sin\beta\sin\alpha=-\boldsymbol{a}\cdot\boldsymbol{\nu}\sin\gamma=-\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}),$$

since $\boldsymbol{\nu}\sin\gamma=\boldsymbol{b}\times\boldsymbol{c}$. Hence:

$$\frac{\sin\mathscr{A}}{\sin\gamma}=-\frac{\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})}{\sin\alpha\sin\beta\sin\gamma}.$$

The last product does not change with cyclic permutation of the vectors and angles, consequently, we have the well-known law of sines:

$$\frac{\sin\mathscr{A}}{\sin\gamma}=\frac{\sin\mathscr{B}}{\sin\beta}=\frac{\sin\mathscr{C}}{\sin\alpha}.$$

§ 1.3. TRANSFORMATION OF ROTATIONS

Finite rotations (transformations) in space can be represented by quaternion algebra in a simple and convenient form. Such a representation is based on the following fundamental theorems [26, 49, 66].

Theorem 1.1. Let $\boldsymbol{\Lambda}$ and $\boldsymbol{R}$ be nonscalar quaternions; then the quantity:

$$\boldsymbol{R}'=\boldsymbol{\Lambda}\circ\boldsymbol{R}\circ\boldsymbol{\Lambda}^{-1} \tag{1.14}$$

is a quaternion, whose norm and scalar part equal the norm and scalar part of $\boldsymbol{R}$. The vector part $\boldsymbol{R}'$ is obtained by rotating $\boldsymbol{R}$ on a cone about the axis vect $\boldsymbol{\Lambda}$ by twice the angle of $\boldsymbol{\Lambda}$. Thus, if:

$$\boldsymbol{\Lambda}=\Lambda\left(\cos\frac{\theta}{2}+\boldsymbol{\xi}\sin\frac{\theta}{2}\right),$$

then vect $\boldsymbol{R}'$ is obtained by rotating vect $\boldsymbol{R}$ about the axis ξ by the angle θ.

Proof. In accordance with the properties of the norm, we have for the norm of $\boldsymbol{R}'$:

$$|\boldsymbol{R}'|=|\boldsymbol{\Lambda}\circ\boldsymbol{R}\circ\boldsymbol{\Lambda}^{-1}|=\|\boldsymbol{\Lambda}\|\|\boldsymbol{R}\|\|\boldsymbol{\Lambda}^{-1}\|=\|\boldsymbol{R}\|\|\boldsymbol{\Lambda}\|\|\boldsymbol{\Lambda}\|^{-1}=|\boldsymbol{R}|. \tag{1.15}$$

By using the property (1.2), we obtain by analogy: /27

$$\operatorname{sqal} R' = \operatorname{sqal}(\Lambda \circ R \circ \Lambda^{-1}) = \operatorname{sqal}(\Lambda^{-1} \circ \Lambda \circ R) = \operatorname{sqal} R. \tag{1.16}$$

Moreover, by writing R in the form $R = r_0 + r$, we obtain formula (1.14) in the following form:

$$\Lambda \circ R \circ \Lambda^{-1} = \Lambda \circ r_0 \circ \Lambda^{-1} + \Lambda \circ r \circ \Lambda^{-1} = r_0 + \Lambda \circ r \circ \Lambda^{-1} = r_0 + r' \tag{1.17}$$

and:

$$R' = r_0 + r',$$

i.e., the operation (1.14) changes only the vector part of the quaternion R:

$$\operatorname{vect}(\Lambda \circ R \circ \Lambda^{-1}) = \Lambda \circ (\operatorname{vect} R) \circ \Lambda^{-1}$$

or:

$$r' = \Lambda \circ r \circ \Lambda^{-1}. \tag{1.18}$$

We determine the nature of the transformation of the vector part of the quaternion (1.17). We represent the quaternion R in the form (1.11):

$$R = |R|(\cos\varphi + \varepsilon \sin\varphi) = R(\cos\varphi + \varepsilon \sin\varphi).$$

Then, in accordance with (1.17), we have:

$$\operatorname{vect} R' = r' = \Lambda \circ (R\varepsilon \sin\varphi) \circ \Lambda^{-1} = R \sin\varphi (\Lambda \circ \varepsilon \circ \Lambda^{-1}) = R\varepsilon' \sin\varphi.$$

In order to show the geometric essence of the transformation of the unit vector ε', we set (Figure 1.7):

$$\varepsilon = i_1 \cos\psi + i_2 \sin\psi,$$

and direct the vector of the quaternion Λ along the i_1 axis, i.e. $\zeta = i_1$, then:

$$\varepsilon' = \Lambda \circ \varepsilon \circ \Lambda^{-1} = (\Lambda \circ i_1 \circ \Lambda^{-1}) \cos\psi + (\Lambda \circ i_2 \circ \Lambda^{-1}) \sin\psi.$$

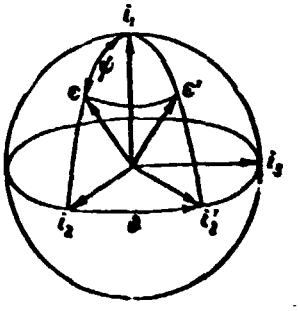

Figure 1.7.

Since vect Λ and $\mathbf{i}_1$ are parallel, then $\Lambda \circ i_1 \circ \Lambda^{-1} = \Lambda \circ \Lambda^{-1} \circ i_1 = i_1$. /28

We have for $\mathbf{i}_2$:

$$\Lambda \circ i_2 \circ \Lambda^{-1} = \left(\cos\frac{\vartheta}{2} + i_1 \sin\frac{\vartheta}{2}\right) \circ i_2 \circ \left(\cos\frac{\vartheta}{2} - i_1 \sin\frac{\vartheta}{2}\right) =$$
$$= \left(i_2 \cos\frac{\vartheta}{2} + i_3 \sin\frac{\vartheta}{2}\right) \circ \left(\cos\frac{\vartheta}{2} - i_1 \sin\frac{\vartheta}{2}\right) =$$
$$= i_2\left(\cos^2\frac{\vartheta}{2} - \sin^2\frac{\vartheta}{2}\right) + i_3\left(2\sin\frac{\vartheta}{2}\sin\frac{\vartheta}{2}\right).$$

i.e. the basis vector $\mathbf{i}_2$ becomes the basis vector:

$$i_2' = i_2 \cos\vartheta + i_3 \sin\vartheta,$$

obtained by rotating the basis vector $\mathbf{i}_2$ about the axis $\mathbf{i}_1$ (i.e., about ζ) by the angle ϑ in the positive direction. Correspondingly, $\varepsilon = i_1 \cos\psi + i_2 \sin\psi$ becomes:

$$\varepsilon' = i_1 \cos\psi + i_2' \sin\psi,$$

i.e., it performs a rotation on a cone with the axis ζ by the angle ϑ.

Thus, it has been shown that with the transformation (1.14) the vector part of R is rotated on a cone about the axis vect Λ.

The operation (1.14) changes only the vector part of the quaternion; for this reason one can consider it as an operation transforming the vector $\mathbf{r}$ into the vector $\mathbf{r}'$, which is written in the form (1.18). Since the norm of the quaternion is not changed by the transformation (1.15), the length of the vector (i.e., its magnitude $r = \sqrt{\|R\| - r_0^2} = \sqrt{r_1^2 + r_2^2 + r_3^2}$) also remains constant:

$$r' = \sqrt{\|R'\| - (r_0')^2} = \sqrt{\|R\| - r_0^2} = r.$$

Thus, it follows that the transformation (1.18) is an orthogonal transformation. According to the Euler theorem [16], any orthogonal transformation is a rotation about some axis. The transformation (1.18) defines at once the axis of this rotation: this axis is vect $\Lambda = \lambda$, since only this axis is not changed by the transformation:

$$\lambda' = \Lambda \circ \lambda \circ \Lambda^{-1} = \Lambda \circ \Lambda^{-1} \circ \lambda = \lambda.$$

In accordance with this we will call the operations (1.18) and (1.14) rotations operations (transformations). /29

We will give the coordinate expression of the transformation (1.14). Let:

$$R = r_0 + r_1 i_1 + r_2 i_2 + r_3 i_3, \quad R' = r'_0 + r'_1 i_1 + r'_2 i_2 + r'_3 i_3$$

and:

$$\Lambda = \lambda_0 + \lambda_1 i_1 + \lambda_2 i_2 + \lambda_3 i_3.$$

Performing the multiplication of (1.14) and equating the terms for the four units, we obtain the expression for the components of the quaternion **R'** in the form of a linear function of the components of **R**:

$$\left.\begin{aligned} r'_0 &= r_0, \\ r'_1 &= \frac{\lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2}{\Lambda^2} r_1 + \frac{2(\lambda_1\lambda_2-\lambda_0\lambda_3)}{\Lambda^2} r_2 + \frac{2(\lambda_1\lambda_3+\lambda_0\lambda_2)}{\Lambda^2} r_3, \\ r'_2 &= \frac{2(\lambda_1\lambda_2+\lambda_0\lambda_3)}{\Lambda^2} r_1 + \frac{\lambda_0^2+\lambda_2^2-\lambda_1^2-\lambda_3^2}{\Lambda^2} r_2 + \frac{2(\lambda_2\lambda_3-\lambda_0\lambda_1)}{\Lambda^2} r_3, \\ r'_3 &= \frac{2(\lambda_1\lambda_3-\lambda_0\lambda_2)}{\Lambda^2} r_1 + \frac{2(\lambda_2\lambda_3+\lambda_0\lambda_1)}{\Lambda^2} r_2 + \frac{\lambda_0^2+\lambda_3^2-\lambda_1^2-\lambda_2^2}{\Lambda^2} r_3. \end{aligned}\right\} \quad (1.19)$$

One can verify that the determinant of the transformation of the components of the vector parts of the quaternion equals one.

When the quaternion Λ specifying the rotation transformation is a normalized quaternion ($\|\Lambda\| = \Lambda^2 = 1$ and $\Lambda^{-1} = \tilde{\Lambda}$), the operations (1.14) and (1.18) are written in the form *:

$$R' = \Lambda \circ R \circ \tilde{\Lambda}, \quad r' = \Lambda \circ r \circ \tilde{\Lambda}. \quad (1.20)$$

---

* When Λ is not a unit quaternion, the transformation (1.20) gives a more general form of a spatial transformation: rotation and dilatation (the magnitude of the latter is determined by Λ^2).

Hence, Λ has obtained the designation of space tensor ("stretcher") and Λ/Λ — versor ("turner"); however, these names are no longer used, since a broader meaning is included in the concept of a tensor. We will call the quantity Λ the magnitude of the quaternion.

Let $\Lambda = \cos\frac{\Phi}{2} + \zeta \sin\frac{\Phi}{2}$ be a normalized quaternion. The form $\Lambda \circ (\ ) \circ \tilde{\Lambda}$ can be considered as a rotation operator specified by the quaternion Λ; by rotation operation defined by Λ we will mean just this operator. We will also say that the quaternion Λ specifies a rotation by the angle Φ, meaning by this the operation (1.20). /30

The operator $\Lambda \circ (\ ) \circ \tilde{\Lambda}$ determines a rotation by an angle Φ. Let us consider the quaternion $(-\Lambda)$; since $(-\tilde{\Lambda}) = -\tilde{\Lambda}$, then the rotations $\Lambda \circ (\ ) \circ \tilde{\Lambda}$ and $(-\Lambda) \circ (\ ) \circ (-\tilde{\Lambda})$ give exactly the same result.

The quaternion:

$$-\Lambda = \cos\left(\pi - \frac{\Phi}{2}\right) + (-\zeta)\sin\left(\pi - \frac{\Phi}{2}\right)$$

specifies a rotation by the angle $2\pi - \Phi$ about the axis $-\zeta$, i.e., in the other direction; it becomes obvious that the result of such a rotation is exactly the same as the result of a rotation by the angle Φ about the axis ζ.

Since the rotation $\tilde{\Lambda} \circ \Lambda \circ (\ ) \circ \tilde{\Lambda} \circ \Lambda = 1 \circ (\ ) \circ 1$, then the rotation $\tilde{\Lambda} \circ (\ ) \circ \Lambda$ specifies a rotation in the opposite direction by the angle Φ, which is obvious besides, since:

$$\tilde{\Lambda} = \cos\frac{\Phi}{2} - \zeta \sin\frac{\Phi}{2} = \cos\left(-\frac{\Phi}{2}\right) + \zeta \sin\left(-\frac{\Phi}{2}\right).$$

Let the normalized quaternion Λ specify a rotation by the angle Φ about the axis λ, and the normalized quaternion M — a rotation by the angle γ about the axis μ. The sequence of rotations Λ and M corresponds to the operator:

$$M \circ \Lambda \circ (\ ) \circ \tilde{\Lambda} \circ \tilde{M} = (M \circ \Lambda) \circ (\ ) \circ (\widetilde{M \circ \Lambda}). \qquad (1.21)$$

Since $M \circ \Lambda$ is also a normalized quaternion, say:

$$M \circ \Lambda = \cos\frac{\Psi}{2} + \nu \sin\frac{\Psi}{2},$$

then the result of the two successive rotations is equivalent to one rotation by the angle ψ about the axis ν; the axis of the resulting rotation and the angle are found from the components of the resulting quaternion according to the rules of their multiplication (1.5).

Thus, the rotation Λ and a following rotation M are equivalent to one rotation $M \circ \Lambda$. In general, the sequence of rotations /31 $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ is equivalent to one rotation $\Lambda_n \circ \Lambda_{n-1} \circ \ldots \circ \Lambda_2 \circ \Lambda_1$.

We will consider some particular examples.

If $\Lambda = \lambda$, i.e. $\theta = 90°$, then the vector:

$$e' = \lambda \circ e \circ \tilde{\lambda} = -\lambda \circ e \circ \lambda$$

is obtained by rotating e by 180° on a cone about the axis λ. Thus, the transformation $-\lambda \circ (\ ) \circ \lambda$ gives a vector symmetric to the first relative to the axis λ (Figure 1.8). The transformation $\lambda \circ (\ ) \circ \lambda$ gives a vector opposite to the vector $-\lambda \circ (\ ) \circ \lambda$, i.e., it can be considered as a mirror reflection relative to the plane perpendicular to λ.

The sequence of two mirror reflections defined by the vectors λ and μ (i.e., the sequence of rotations $\lambda \circ (\ ) \circ \lambda$ and $\mu \circ (\ ) \circ \mu$) is equivalent to one operator:

$$\mu \circ \lambda \circ (\ ) \circ \lambda \circ \mu = \mu \circ \lambda \circ (\ ) \circ (\mu \circ \lambda)^{-1},$$

specifying the rotation. But since $\mu \circ \lambda = -\mu \cdot \lambda - \lambda \times \mu$, then:

$$\mu \circ \lambda = -\cos\varphi - b \sin\varphi = -(\cos\varphi + b \sin\varphi),$$

where φ is the angle between the vectors λ and μ.

Hence, it follows that a sequence of two mirror reflections relative to two planes is equivalent to a rotation about the line of intersection of the planes by twice the angle between them.

Recalling the representation of the unit quaternion as an arc of a great circle on the unit sphere, whose pole is the vector of the quaternion, one can see that the rotation operation gives a rotation defined by twice the angle (i.e., twice the arc) about this same pole.

Let us now consider a spherical polygon; let its sides, going around sequentially (cyclically), represent the quaternions /32
$\Lambda_1, \Lambda_2, \ldots, \Lambda_n$; in accordance with the results of paragraph 1.2, we have:

$$\Lambda_n \circ \Lambda_{n-1} \circ \ldots \circ \Lambda_1 = 1.$$

Since $(\overline{\Lambda_n \circ \Lambda_{n-1} \circ \ldots \circ \Lambda_1}) = 1$, then the operator:

$$\Lambda_n \circ \Lambda_{n-1} \circ \ldots \circ \Lambda_1 \circ (\ ) \circ \tilde{\Lambda}_1 \circ \tilde{\Lambda}_2 \circ \ldots \circ \tilde{\Lambda}_n = 1 \circ (\ ) \circ 1,$$

i.e., such a sequence of rotations $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ gives the original state. Hence, the Hamilton-Donkin theorem, in particular, follows:

If $\mathscr{ABC}$ is some spherical triangle, then the three sequential rotations defined by the arcs $2\mathscr{BC}$, $2\mathscr{CA}$ and $2\mathscr{AB}$ (i.e., relative to their polar axes) brings the body to its original state.

§ 1.4. SPHERICAL GEOMETRY

The multiplication rules for the hypercomplex units are obtained in a rather natural manner: the squares of the imaginary units equal -1 and the product $i_1 \circ i_2$ can be set in a natural manner equal to i_3; the remaining multiplication rules are obtained from the associative property of multiplication *.

---

* Actually, from these equalities we obtain by multiplication:

$$i_1 \circ (i_1 \circ i_2) = i_1 \circ i_3 = (i_1 \circ i_1) \circ i_2 = -i_2,$$
$$(i_1 \circ i_2) \circ i_2 = i_3 \circ i_2 = i_1 \circ (i_2 \circ i_2) = -i_1.$$

Multiplying these two equalities:

$$(i_1 \circ i_3) \circ (i_3 \circ i_2) = i_2 \circ i_1 = i_1 \circ (i_3 \circ i_3) \circ i_2 = -i_1 \circ i_2 = -i_3,$$
$$(i_3 \circ i_2) \circ (i_2 \circ i_1) = i_1 \circ i_3 = i_3 \circ (i_2 \circ i_2) \circ i_1 = -i_1 \circ i_3 \text{ etc.}$$

one can verify that all the multiplication rules of the hypercomplex units are satisfied.

These rules define numbers, which have a double meaning: on one hand, they are some geometric forms (vectors in four-dimensional space), and on the other, they are transformation operators in three-dimensional space. In order to illustrate these properties of quaternions, we will now consider some problems of spherical geometry, which contributed historically to the discovery of quaternions.

Geometry on a sphere resembles in many respects geometry on /33
a plane. Thus, if $\mathscr{A}$ and $\mathscr{B}$ are two points on a sphere, then by analogy to the segment on a plane one can consider the quantity $\mathscr{AB}=\mathscr{B}-\mathscr{A}$, which is understood as the distance between the points $\mathscr{A}$ and $\mathscr{B}$, i.e., the arc of the great circle running from point $\mathscr{A}$ to point $\mathscr{B}$.

The concept of the difference of two points as the arc joining them also includes the concept of the sum; it is obvious that:

$$(\mathscr{B}-\mathscr{A})+\mathscr{A}=\mathscr{B},$$

hence, it follows that:

$$\mathscr{AB}+\mathscr{A}=\mathscr{B}. \tag{1.22}$$

The meaning of the latter operation can be interpreted as: the arc $\mathscr{AB}$ shifts the point $\mathscr{A}$ to the point $\mathscr{B}$. A new meaning is already being attributed to the arc here, that is, the meaning of an operator; for this reason the arc is also called the versor.

This definition attributes a definite direction to the arc $\mathscr{AB}$: from point $\mathscr{A}$ to point $\mathscr{B}$. By the arc $\mathscr{BA}$ is understood the arc of the great circle running from point $\mathscr{B}$ to point $\mathscr{A}$, i.e., the arc in the opposite direction *: since $\mathscr{BA}=\mathscr{A}-\mathscr{B}$ and $\mathscr{AB}+\mathscr{BA}=\mathscr{B}-\mathscr{A}+\mathscr{A}-\mathscr{B}=0$, then $\mathscr{BA}=-\mathscr{AB}$. This definition of the difference of points is sufficient to specify the addition and subtraction of arcs on the plane.

* It is obvious that there are two arcs running from point $\mathscr{A}$ to point $\mathscr{B}$ on the great circle. For now, we will not make a distinction between them.

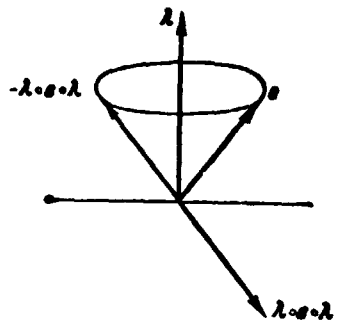

Figure 1.8.

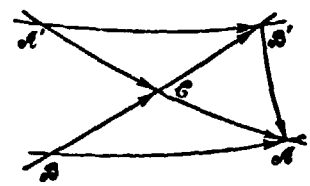

Figure 1.9.

Let us consider the equation $(\mathscr{A}-\mathscr{C})+(\mathscr{C}-\mathscr{B})=(\mathscr{A}-\mathscr{B})$, which is written by definition as:

$$\mathscr{CA}+\mathscr{BC}=\mathscr{BA}, \tag{1.23}$$

i.e., the sum of the two sides of a spherical triangle equals its third side (Figure 1.9).

Two arcs will be considered equal if and only if they lie on the same great circle, and their lengths and directions are equal. Equal arcs can be superimposed by moving along the great circle on which they are located. By noting that by this definition $\mathscr{BC}=\mathscr{CB}$ and $\mathscr{A}'\mathscr{C}=\mathscr{CA}$ (cf. Figure 1.9), we obtain by interchanging the terms in equation (1.23): /34

$$\mathscr{BC}+\mathscr{CA}=\mathscr{CB}'+\mathscr{A}'\mathscr{C}=\mathscr{A}'\mathscr{B}',$$

i.e., we obtain a new arc not equaling the arc $\mathscr{BA}$. Hence, it follows that the addition of arcs on a sphere is noncommutative. The addition of arcs is commutative only if they lie on one arc of a great circle: then the addition reduces to a simple arithmetic sum.

The subtraction of arcs can be obtained by addition with a change of sign of one term. Thus, let us consider the difference $\mathscr{CA}-\mathscr{BC}$ (cf. Figure 1.9); since $\mathscr{BC}=-\mathscr{CB}$, then:

$$\mathscr{CA}-\mathscr{BC}=\mathscr{CA}+\mathscr{CB}=\mathscr{CA}+\mathscr{B}'\mathscr{C}=\mathscr{B}'\mathscr{A}.$$

We set in correspondence to each arc $\mathscr{AB}$ its pole from which the rotation from point $\mathscr{A}$ to point $\mathscr{B}$ is viewed as proceeding counterclockwise. This establishes the distinction between the two paths from $\mathscr{A}$ to $\mathscr{B}$ (by the angle θ and $2\pi-\theta$), since its own pole will correspond to each path.

In spherical geometry there are two series of forms for which all the theorems are valid: the arcs of great circles and their poles. The correspondence of a pole to the arc of its equator, i.e., the correspondence of a point to the arc, appears so adequate that one should introduce in a certain sense their equaltiy.

Actually, the definition of an arc as the difference of two points was given previously. However, the operation of point addition has not yet been properly defined. In order to remain within the framework of the statement that the "additivity" of points gives an arc, one should make a new premise. We set the point and arc mutually equivalent, i.e. to each point as a pole we equate the positive arc of its equator of length $\pi/2$ (quadrant). We will show that the rules considered above remain valid with the premise.

Actually, $\mathscr{B}-\mathscr{A}=\mathscr{AB}$, $\mathscr{B}=\mathscr{ED}$ and $\mathscr{A}=\mathscr{EC}$, i.e., the points /35
equal their quadrant-versors (Figure 1.10). We have:

$$\mathscr{B}-\mathscr{A}=\mathscr{ED}-\mathscr{EC}=\mathscr{ED}+\mathscr{CE}=\mathscr{CD},$$

and actually, $\mathscr{CD}=\mathscr{AB}$, since they lie on the same great circle and the lengths of the arcs $\mathscr{BD}$ and $\mathscr{AC}$ equal $\pi/2$. The equality $(\mathscr{B}-\mathscr{A})+\mathscr{A}=\mathscr{B}$ is also valid, since:

$$\mathscr{AB}=\mathscr{CD} \quad \text{and} \quad \mathscr{A}=\mathscr{EC},$$

from which it follows that:

$$(\mathscr{B}-\mathscr{A})+\mathscr{A}=\mathscr{AB}+\mathscr{A}=$$
$$=\mathscr{CD}+\mathscr{EC}=\mathscr{ED}=\mathscr{B}.$$

To any point $\mathscr{A}$ on the sphere corresponds its antipode $\mathscr{A}'$, and a versor of the opposite direction corresponds to the antipode $\mathscr{A}'$, i.e.:

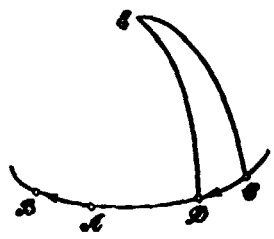

Figure 1.10.

$$\mathcal{A}+\mathcal{A}'=0 \quad \text{and} \quad \mathcal{A}=-\mathcal{A}';$$

It is not difficult to see that these equalities are also valid for the quadrant-arcs of these points. Hence, one can convert the sum of two points into their difference and thereby define it. Let us consider, for example, the sum:

$$\mathcal{B}+\mathcal{A}=\mathcal{B}-\mathcal{A}'=\mathcal{A}'\mathcal{B}=(\mathcal{A}\mathcal{B}+\mathcal{A}\mathcal{A}')=\mathcal{A}\mathcal{B}+\pi, \tag{1.24}$$

where π denotes the arc $\mathcal{A}\mathcal{A}'$ equaling half a great circle. By interchanging $\mathcal{B}$ and $\mathcal{A}$, we have:

$$\mathcal{A}+\mathcal{B}=\mathcal{B}\mathcal{A}+\pi,$$

but since $+\pi$ equals $-\pi$ and an arc is unchanged by the addition of 2π, we obtain:

$$\mathcal{A}+\mathcal{B}=-(\mathcal{A}\mathcal{B}+\pi)=-(\mathcal{B}+\mathcal{A}).$$

So, the sum of two points is also an arc (versor) not equaling a quadrant, generally speaking; thus, the sum (as well as the difference) does not equal a point, generally speaking. The sum changes sign with a permutation of the terms.

There is a unique case when the sum and difference of points /36 gives a point. Let $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ be three points of a sphere such that the arcs $\mathcal{I}_1\mathcal{I}_2$, $\mathcal{I}_2\mathcal{I}_3$ and $\mathcal{I}_3\mathcal{I}_1$ are quadrants with positive poles at points $\mathcal{I}_3$, $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively (Figure 1.11).

Then:

$$\mathcal{I}_1+\mathcal{I}_2=\mathcal{I}_2\mathcal{I}_1+\pi=\mathcal{I}_1\mathcal{I}_2.$$

But since $\mathcal{I}_1\mathcal{I}_2=\mathcal{I}_2-\mathcal{I}_1=\mathcal{I}_3$, by definition, then $\mathcal{I}_1+\mathcal{I}_2=\mathcal{I}_3$. From this relation it follows that the sum or difference of two points determines another point only if the angle between them equals $\pi/2$. By permuting $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$, cyclically, we obtain:

$$\mathcal{I}_1+\mathcal{I}_2=\mathcal{I}_3, \quad \mathcal{I}_2+\mathcal{I}_3=\mathcal{I}_1, \quad \mathcal{I}_3+\mathcal{I}_1=\mathcal{I}_2. \tag{1.25}$$

Since the quadrants for the antipodes change direction, we have:

$$\mathcal{I}_2+\mathcal{I}_1=\mathcal{I}_3', \quad \mathcal{I}_3+\mathcal{I}_2=\mathcal{I}_1', \quad \mathcal{I}_1+\mathcal{I}_3=\mathcal{I}_2'. \tag{1.26}$$

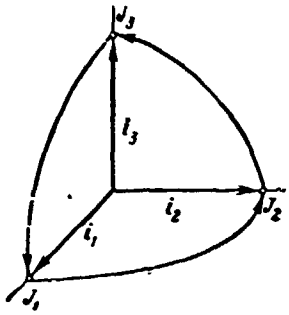

Figure 1.11.

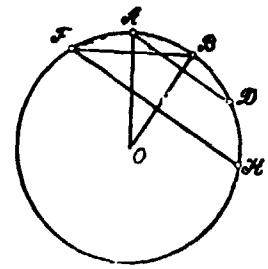

Figure 1.12.

One can also find a double meaning of point and versor in these formulae: besides the direct geometric form, the point (versor) can be understood as a rotation operator. Thus, the first term is an operator shifting the second term into the resultant: $\mathcal{I}_1$ rotates the point $\mathcal{I}_2$ by one quarter of a circle, after which it coincides with $\mathcal{I}_3$, etc.

One can now prove a theorem about rotation.

Let five points $\mathcal{F}, \mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{H}$, separated by equal arcs be given on a great circle of the sphere (Figure 1.12). We assume that there are two coincident spheres, one movable, the other fixed. We turn the movable sphere about the axis $O\mathcal{A}$ by the angle π; then point $\mathcal{B}$ coincides with point $\mathcal{F}$. Now we turn the movable sphere about the axis $O\mathcal{B}$ by the angle π; then point $\mathcal{A}$ coincides with point $\mathcal{D}$, and point $\mathcal{F}$ (i.e. $\mathcal{B}$) — with point $\mathcal{H}$. In this final position the arc $\mathcal{AB}$ appears to have been shiften along the arc of the great circle to the position of $\mathcal{DH}$, i.e., shifted by the arc $\mathcal{AD}=2\mathcal{AB}$. The first rotation is a rotation by twice the quadrant $\mathcal{A}$, i.e. by $2\mathcal{A}$, and the second — correspondingly by $2\mathcal{B}$. /37

Hence, it follows that the sum of the rotations $2\mathcal{B}+2\mathcal{A}$ is equivalent to the rotation by the arc $2\mathcal{AB}$, i.e.:

$$2\mathcal{B}+2\mathcal{A}=2\mathcal{AB}$$

(the latter also follows from (1.24), since a rotation by the angle 2π leads to the original position).

One can show in precisely the same way that:

$$2\mathcal{C}+2\mathcal{B}=2\mathcal{BC},$$

and consequently, by completing two such rotations sequentially, we obtain:

$$2\mathcal{C}+2\mathcal{B}+2\mathcal{B}+2\mathcal{A}=2\mathcal{BC}+2\mathcal{AB}.$$

Obviously, the rotation $2\mathcal{B}+2\mathcal{B}$ leads to the original state; in addition, $2\mathcal{C}+2\mathcal{A}=2\mathcal{AC}$. Hence:

$$2\mathcal{AC}=2\mathcal{BC}+2\mathcal{AB}, \tag{1.27}$$

i.e., the resultant of the successive rotations of the sphere by the doubled arc $2\mathcal{AB}$ about its positive pole and by the doubled arc $2\mathcal{BC}$ about its positive pole is always equivalent to a rotation of the sphere about the third arc of the spherical triangle $\mathcal{ABC}$, that is, by the doubled arc $2\mathcal{AC}$ about the positive pole of the arc $\mathcal{AC}$. Consequently, if the relation π, σ, τ exists between three versors $\pi+\sigma=\tau$, then the condition:

$$2\pi+2\sigma=2\tau,$$

will always take place with respect to rotation. The associativity of rotation addition follows from the consideration that this equivalence is independent of the initial position of the sphere. /38

The obtained material is sufficient to proceed to the study of operations on vectors in space. We associate a radius vector with each point. It is necessary to establish the correspondence of the addition operations found for versors and points with the vector operations. Since versor addition is noncommutative, only the operation of vector multiplication could correspond to it;

for this reason, we will call spherical addition multiplication, and subtraction — division.

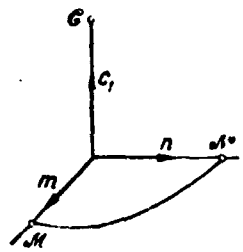

Figure 1.13.

We will denote the vectors of the points $\mathscr{A}, \mathscr{B}, \mathscr{C}$, lying on the unit sphere by $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$. One can show that all the basic statements specifying the effects on points are maintained for the presented definition of the effects on vectors. By the fraction of two vectors $\mathbf{b}_1/\mathbf{a}_1 = b_1 \circ a_1^{-1}$ we will understand the difference of points $\mathscr{B}-\mathscr{A}$, i.e., the versor $\mathscr{A}\mathscr{B}$. Obviously, the spherical triangle relation (1.23) is preserved: the relation $(\mathscr{A}-\mathscr{C})+(\mathscr{C}-\mathscr{B})=\mathscr{C}\mathscr{A}+\mathscr{B}\mathscr{C}=\mathscr{B}\mathscr{A}$ corresponds to the equality:

$$a_1 \circ c_1^{-1} \circ c_1 \circ b_1^{-1} = a_1 \circ b_1^{-1},$$

i.e., the product of two fractions gives a fraction of the same type.

Since the versor $\mathscr{M}\mathscr{N}=\mathscr{N}-\mathscr{M}$, corresponds to the point $\mathscr{C}$, for example, then to each vector $\mathbf{c}_1$ there corresponds a fraction $n \circ m^{-1}$, i.e., $c_1 = n \circ m^{-1}$. Obviously, the vector $\mathbf{c}_1'$ corresponding to the antipode $\mathscr{C}'$ is opposite to the vector $\mathbf{c}_1$, i.e.:

$$c_1' = -c_1.$$

On the other hand, since the point $\mathscr{C}'$ corresponds to the versor $\mathscr{N}\mathscr{M}=\mathscr{M}-\mathscr{N}$, then $c_1' = m \circ n^{-1}$. Hence, it follows that $c_1 \circ c_1' = 1$, as it should, since for the versors $\mathscr{C}+\mathscr{C}'=0$.

The obtained relation shows that the vector $\mathbf{c}_1'$ opposite to the vector $\mathbf{c}_1$ satisfies the condition $c_1' = c_1^{-1}$, because only in this case $c_1 \circ c_1' = 1$. But, on the other hand, $c_1' = -c_1$, from which it follows that $c_1 \circ (-c_1) = 1$ and $c_1 \circ c_1 = -1$, i.e., the square of any vector (unit) equals -1. /39

These statements permit one to obtain in accordance with (1.25) and (1.26) the following relations for the system of three orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ passing through the points $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ (cf. Figure 1.11):

$$\left.\begin{array}{lll} i_1 \circ i_1 = -1, & i_2 \circ i_2 = -1, & i_3 \circ i_3 = -1, \\ i_1 \circ i_2 = i_3, & i_2 \circ i_3 = i_1, & i_3 \circ i_1 = i_2, \\ i_2 \circ i_1 = i_3' = -i_3, & i_3 \circ i_2 = i_1' = -i_1, & i_1 \circ i_3 = i_2' = -i_2, \end{array}\right\} \qquad (1.28)$$

where $\mathbf{i}_j'$ are the vectors in the opposite direction.

The rules found for multiplying the unit vectors can be transferred to the multiplication of arbitrary vectors in space. For this we represent the arbitrary vector $\mathbf{a}$ in the form:

$$a = \alpha a_1,$$

where $\mathbf{a}_1$ is the unit vector and α is a number (scalar), which we call the vector tensor. Let us consider the fraction of the two vectors $a = \alpha a_1$ and $b = \beta b_1$, which we denote by Λ:

$$\Lambda = \frac{a}{b} = a \circ b^{-1} = \frac{\alpha}{\beta} a_1 \circ b_1^{-1}. \qquad (1.29)$$

The first factor of the obtained fraction α/β is called the tensor, and the second $a_1 \circ b_1^{-1}$ — the versor. The fraction Λ depends on the ratio of the lengths of the vectors, on the angle between them and on the position of the plane. Since the position of the plane is determined by two parameters, then four parameters in all are needed to define Λ. For this reason Λ is called a quaternion.

The orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ (1.28) are thusly quaternions also. Their multiplication rules reveal the same doubled meaning characteristic of quaternions: on one hand, they are some geometric forms, on the other, they are operators. Thus, the first factor in the product $i_1 \circ i_2$ can be considered as an operator acting on the vector $\mathbf{i}_2$ and converting it into the vector /40 $\mathbf{i}_3$. Since the quaternion (1.29) is defined as a fraction, then if the angle between the vectors **a** and **b** equals the angle between the

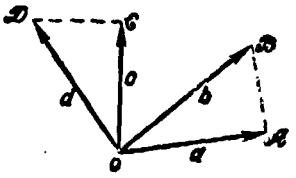

Figure 1.14.

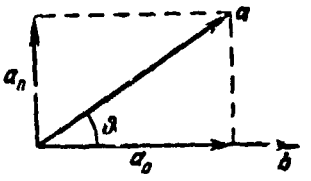

Figure 1.15.

vectors **c** and **d**, the ratios **ab/** and **c/d** are equal and, in addition, the planes formed by the vector pairs **a**, **b** and **c**, **d** coincide (Figure 1.14), then their ratios:

$$a \circ b^{-1} = c \circ d^{-1}$$

form the same quaternion. Thus, a quaternion is determined by the plane formed by the vectors **a**, **b** and the angle between them.

We note that since a point (vector) corresponds to the versor $\pi/2$, then the fraction of two mutually perpendicular segments is a vector. On the other hand, if the vectors **a** and **b** are collinear, then their fraction is a simple real number.

Let us consider the case of [illegible] arbitrary angle between the vectors a and b (Figure 1.15). The vector **a** can be resolved into the sum $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_n$, then:

$$\Lambda = (a_0 + a_n) \circ b^{-1} = a_0 \circ b^{-1} + a_n \circ b^{-1}$$

will represent the sum of a real number equaling the ratio of the magnitudes a_0/b and a vector. Hence, it follows that each quaternion can be uniquely resolved into a scalar and a vector part. It is not difficult to see that the scalar part of (1.29) equals:

$$\frac{a_0}{b} = \frac{a \cos \vartheta}{b} = \frac{a}{\beta} \cos \vartheta,$$

and the magnitude of the vector part equals $\frac{a}{\beta} \sin \vartheta$ by analogy.

If the vector of the quaternion is further resolved along the orthogonal unit axes $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, then we arrive at the representation of the quaternion as the hypergeometric number, with which this chapter began.

CHAPTER 2

THEORY OF FINITE RIGID BODY ROTATIONS

Orthogonal transformations, which are the mathematical form for describing rigid body rotation [16, 44, 66], will be considered in this chapter.

Arbitrary finite motion of a rigid body can be resolved into translational motion linked with some arbitrarily selected point and rotation relative to this point. In accordance with this, of the six independent coordinates describing rigid body motion, three coordinates specify the translation of some fixed point of the body in space, and three coordinates characterize its rotation. We will be interested only in the rotation of the rigid body relative to the fixed point. This problem is an independent problem in mechanics: its solution is required when studying any motion of a body, whose finite dimensions are to be taken into account. In the majority of practical cases, the rotational motion does not depend on the translational motion of the rigid body. Thus, for example, when the fixed point is the center of mass, the body is in free motion in an arbitrary force field.

For the solution of any dynamic or kinematic problem, where the rigid body cannot be considered as a point, one requires a rational description for the kinematics of the rotational motion. The essential role in this is played by the choice of the kinematic parameters corresponding to the three rotational degrees of freedom of the rigid body motion. A number of kinematic parameters used for describing rigid body motion are known: the direction cosines, the Euler and Krylov angles, the Cayley-Klein parameters, the Rodriguez - Hamilton parameters

[19, 31, 44]. Among the most convenient parameters are the Rodriguez - Hamilton parameters, which are quaternion components. These parameters are not degenerate for any position of the rigid body (i.e., the parameters themselves as well as /42 their rates of change do not become infinite), in contrast to the Euler angles. There are four Rodriguez - Hamilton parameters, i.e., there is one coupling equation, in contrast to the six coupling equations for the direction cosines. On the other hand, these parameters characterize the most natural method for specifying the position of the rigid body with the help of the plane of rotation.

The finite rotation of a rigid body (i.e., the motion of a body having a fixed point) has the peculiarity that it always leaves one axis fixed in space. This fact is established by the well-known Euler theorem, according to which: any rigid body rotation is equivalent to a plane rotation about some axis and can be specified as a finite rotation about this axis or as a finite rotation vector directed along the Euler rotation axis and having a length depending on the rotation angle. Plane rotation has the same angle independent of the center of rotation; it is essential that the axes of rotation remain parallel to each other. In this sense the finite rotation vector is an invariant of the arbitrary motion of the body. In particular, an arbitrary motion can be resolved into two translational motions - along the axis of the rotation vector and in the plane of rotation - and the inherent rotation. By the choice of the axis (or center) of plane rotation, one can reduce the translational motion in the plane of rotation to zero (i.e., combine it with the rotation; then the axis of rotation becomes the so-called center of rotation); the arbitrary motion of the rigid body is reduced in this case to helical motion [19].

The finite rotation vector defines the Rodriguez - Hamilton parameters (as well as the Cayley - Klein parameters).

The application of such a type of kinematic parameters leads in a natural manner to the use of quaternions. Quaternions have a structure permitting one to describe in a most convenient manner all the operations linked with the description and study of rigid body motion. /43

Orthogonal transformations which are finite rotations according to the Euler theorem will be considered in this chapter. The isomorphic correspondence of the orthogonal transformation operation and the quaternion multiplication operation will be established. The application of quaternions permits the most complete study of the theory of finite rotations. In particular, all the known results of the theory of finite rotations [31] are obtained in the form of quaternion multiplication operations. The use of quaternions permits one to obtain a number of new results: the transformation (rotation) addition theorem for the Rodriguez-Hamilton parameters, the generalized theorem for finite rotation permutativability, etc.

On the other hand, the forms obtained for the quaternion operations corresponding to the orthogonal transformations have a sufficiently general form and are repeated when writing the analogous operations in any parameters. The study presented below of the relation of the Rodriguez-Hamilton parameters with the other kinematic parameters rather clearly illustrates this fact.

The structure developed in this chapter permits one to represent in a convenient form all the operations related to orthogonal transformations.

§ 2.1. ORTHOGONAL TRANSFORMATIONS

Rigid body rotation can be described by various methods. The most general is the method in which the rigid body orientation is determined by the orientation of an orthogonal (Cartesian) coordinate system connected to the body.

Let us assume that a rectangular coordinate system **E** having the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is rigidly bound to a rigid body. The motion of the body will be considered relative to a rectangular coordinate system **I** with the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$. The coordinate systems **E** and **I** will be called the body and fixed bases, respectively. Let us further assume that the origins of both bases are located at the same point about which the rigid body motion occurs. / 44

The position of any axis $\mathbf{e}_k$ of the basis **E** in the coordinate system **I** is determined by the three direction cosines formed by this axis with the axes of $\mathbf{i}_j$ (k,j = 1, 2, 3)

$$a_{k1} = \mathbf{e}_k \cdot \mathbf{i}_1, \quad a_{k2} = \mathbf{e}_k \cdot \mathbf{i}_2, \quad a_{k3} = \mathbf{e}_k \cdot \mathbf{i}_3.$$

The system of nine direction cosines completely determines the orientation of the body basis **E** relative to the reference coordinate system. We place these cosines in a 3x3 I matrix called the direction cosine matrix:

$$\mathbf{A} = \| a_{kj} \| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \| \mathbf{e}_k \cdot \mathbf{i}_j \|. \qquad (2.1)$$

Any basis vector of the body system **E** can be expressed through the basis vectors of the basis **I** using the relations*:

$$e_k = (e_k \cdot i_1)\, i_1 + (e_k \cdot i_2)\, i_2 + (e_k \cdot i_3)\, i_3 = \sum_j a_{kj} i_j. \qquad (2.2)$$

This relation is one of the methods for specifying rigid body motion, i.e., by transforming one coordinate system into another.

Let us now consider how the direction cosines permit one to express the components of some invariant vector in the coordinate system **E**, if its components are known in the system **I**. Let **r** be an invariant vector, and r and r'_k be its projections onto the bases **I** and **E**, respectively. It is obvious that:

$$r = r_1 i_1 + r_2 i_2 + r_3 i_3 = r'_1 e_1 + r'_2 e_2 + r'_3 e_3. \qquad /45$$

Hence, it is easy to find that:

$$r'_k = r \cdot e_k = r_1 i_1 \cdot e_k + r_2 i_2 \cdot e_k + r_3 i_3 \cdot e_k,$$

i.e.:

$$r'_k = \sum_{j=1}^{3} a_{kj} r_j. \qquad (2.3)$$

It is obvious that the transformation (2.3) can define the transformation of the coordinate system. This is another way of specifying the transformation of bases, when the change of the components of the invariant vector specified by the relations (2.3) is described identically to the relation (2.2).

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* We note for reference that the elements α_{kj} defined by the relations (2.1) are the covariant components of the vector e_k in the basis **I**. The resolution of the vector e_k onto the unit vectors of the basis is specified by the contravariant components of e_k in the basis **I**. The covariant and contravariant components **E** and **I** coincide for the orthogonal coordinate systems [28], thanks to which formula (2.2) is possible.

The relations (2.3) are the linear transformation equations defined by the matrix $\|a_{ij}\|$. Since this transformation describes rigid body motion, the matrix (2.1) is an orthogonal transformation matrix. The elements a_{ij} satisfy the well-known orthogonality conditions:

$$\sum_{i=1}^{3} a_{ij}a_{ik} = \sum_{i=1}^{3} a_{ji}a_{ki} = \delta_{jk} = \begin{cases} 1 \text{ with } j=k, \\ 0 \text{ with } j \neq k \end{cases}. \qquad (2.4)$$

The orthogonality conditions are a reflection of the property of a rigid body to maintain fixed distances between any of its points. The invariance of the length of the vector **r** indicates that the equality:

$$\sum_{i=1}^{3} r_i^2 = \sum_{i=1}^{3} (r_i')^2 = \\ = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} a_{ij}r_j \right) \left(\sum_{k=1}^{3} a_{ik}r_k \right) = \sum_{j,k=1}^{3} \left(\sum_{i=1}^{3} a_{ij}a_{ik} \right) r_j r_k, \qquad (2.5)$$

must be satisfied, from which the first condition (2.4) also follows.

Let us consider the inverse transformation converting the basis **E** into the basis **I**; it will be defined by the matrix $\mathbf{A}^{-1} = \|a'_{ij}\|$, the inverse of the matrix **A**. By analogy to the relations (2.2) and (2.4), the inverse transformations can be written in the form of equalities relating the basis vectors of the bases /46 **E** and **I** or in the form of relations between the projections of an invariant vector onto these bases:

$$\left. \begin{aligned} i_k &= \sum_{j=1}^{3} a'_{kj}e_j, \\ r_j &= \sum_{k=1}^{3} a'_{jk}r'_k. \end{aligned} \right\} \qquad (2.6)$$

By repeating the derivation of (2.5), one finds that the elements of the inverse matrix must also satisfy the orthogonality conditions:

$$\sum_{i=1}^{3} a'_{ij}a'_{ik} = \delta_{jk}. \qquad (2.7)$$

On the other hand, let us consider the following representation of the vector **r**:

$$r=\sum_{i=1}^{3} r_i i_i=\sum_{j=1}^{3} r'_j e_j=\sum_{j=1}^{3} r'_j\left(\sum_{k=1}^{3} a_{jk} i_k\right).$$

We change the order of the summation in the sum obtained by first summing over the second index and then over the first:

$$r=\sum_{k=1}^{3}\left(\sum_{j=1}^{3} a_{kj} r'_j\right) i_k.$$

i.e., first, the matrix **A** is transposed, and second, we obtain:

$$r_k=\sum_{j=1}^{3} a_{kj} r'_j \qquad (2.8$$

By comparing the obtained relation (2.8) with the equalities (2.6), one can see that:

$$a'_{jk}=a_{kj}, \qquad (2.9)$$

i.e., the elements of the inverse matrix equal the elements of the transposed matrix. It is obvious that the equality (2.7) then changes into the second orthogonality condition (2.4). In addition, the conditions (2.9) are orthogonality conditions themselves, completely equivalent* to the conditions (2.4). By denoting the transposed matrix $\|a_{kj}\|=\mathbf{A}^{\mathrm{T}}$ and the inverse $\|a'_{jk}\|=\mathbf{A}^{-1}$, / 47 we write the condition (2.9) in the form:

$$\mathbf{A}^{-1}=\mathbf{A}^{\mathrm{T}}. \qquad (2.10)$$

The orthogonality conditions also follow directly from the relations (2.2) and (2.6), if one takes the fact into account that the following equalities are satisfied for an orthonormal basis:

$$i_j \cdot i_k=\delta_{jk}, \quad e_i \cdot e_j=\delta_{ij}. \qquad (2.11)$$

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* This can be verified directly by calculating the elements of the inverse matrix, which are equal to the corresponding complementary minors. The equalities (2.9) are satisfied when the equalities (2.4) are true, and conversely.

By substituting the expressions for the unit vectors (2.2) or (2.6) into these equalities, we obtain the conditions (2.4) directly, which obviously is a reflection of the orthogonality conditions.

We will use the matrix symbolism for convenience in specifying the transformations. We introduce the column matrices composed of the unit vectors $\mathbf{i}_j$ and $\mathbf{e}_k$, which we will denote by the same symbols as for the bases, i.e.:

$$\mathbf{I}=\begin{vmatrix} i_1 \\ i_2 \\ i_3 \end{vmatrix}, \quad \mathbf{E}=\begin{vmatrix} e_1 \\ e_2 \\ e_3 \end{vmatrix}. \tag{2.12}$$

By using the matrix multiplication operation, one can write the transformation (2.2) in the form:

$$\mathbf{E}=\mathbf{AI}, \quad \mathbf{I}=\mathbf{A}^{-1}\mathbf{E}=\mathbf{A}^{\mathrm{T}}\mathbf{E}. \tag{2.13}$$

We also introduce the column matrices composed of the components of the vector $\mathbf{r}$ in some basis, i.e., of the quantities $\mathbf{r}_i$ and $\mathbf{r}_j$:

$$\mathbf{R}=\begin{vmatrix} r_1 \\ r_2 \\ r_3 \end{vmatrix}, \quad \mathbf{R}'=\begin{vmatrix} r_1' \\ r_2' \\ r_3' \end{vmatrix}.$$

The representation (2.3) can be written in the matrix form:

$$r=\mathbf{R}'^{\mathrm{T}}\mathbf{E}=\mathbf{E}^{\mathrm{T}}\mathbf{R}'=\mathbf{R}'^{\mathrm{T}}\mathbf{I}=\mathbf{I}^{\mathrm{T}}\mathbf{R}, \tag{2.14}$$

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and these relations can easily be obtained with the use of the introduced matrix symbols*:

$$\mathbf{E}^{\mathrm{T}}\mathbf{R}'=(\mathbf{AI})^{\mathrm{T}}\mathbf{R}'=\mathbf{I}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{R}'=\mathbf{I}^{\mathrm{T}}\mathbf{R},$$

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* The following matrix property is also used: the transpose of the product of two matrices equals the product of the transposed matrices in reverse order.

from which

$$\mathbf{R}=\mathbf{A}^{\mathrm{T}}\mathbf{R}', \quad \mathbf{R}'=\mathbf{A}\mathbf{R} \tag{2.15}$$

(here $\mathbf{A}^{\mathrm{T}}\mathbf{A}=\mathbf{A}\mathbf{A}^{\mathrm{T}}=\mathbf{1}$, where $\mathbf{1}=\|\delta_{ij}\|$ is the unit matrix).

The orthogonal transformation in the form (2.13) and (2.15) is defined by the matrix **A**, which can be considered as an operator transforming **I** into **E** (for the first method for specifying rotation), or as an operator specifying the transition between the components of an invariant vector with a basis transformation (for the second method for specifying rotation transformation). Obviously, both these methods are completely equivalent, and it is sufficient to specify rotation by either of these methods.

However, there is another method for specifying rigid body rotation, which has another interpretation of an orthogonal transformation.

Let us consider a vector, whose position is not changed relative to the basis being transformed, i.e., a vector which rotates with the coordinate system. Obviously, such a vector changes its position in space as a result of the rotation of the basis; in accordance with this, the vector **r** is converted to the new vector **r'** with the basis transformation. The vector **r'** can be determined, since the components of the vector **r** in the original basis and the components of the vector **r'** in the transformed basis are the same:

$$r=\sum_{j=1}^{3} r_j i_j=\mathbf{R}^{\mathrm{T}}\mathbf{I}, \quad r'=\sum_{j=1}^{3} r_j e_j=\mathbf{R}^{\mathrm{T}}\mathbf{E}. \tag{2.16}$$

By using the relation (2.13), one can find the transformed vector:

$$r'=\mathbf{R}^{\mathrm{T}}\mathbf{E}=\mathbf{R}^{\mathrm{T}}\mathbf{A}\mathbf{I}=(\mathbf{A}^{\mathrm{T}}\mathbf{R})^{\mathrm{T}}\mathbf{I}=\mathbf{R}'^{\mathrm{T}}\mathbf{I},$$

Where **R'** is the column matrix of the components of the vector **r'** in the basis of **I**. It follows from the equality obtained that: /49

$$\mathbf{R}' = \mathbf{A}^{\mathrm{T}}\mathbf{R}, \tag{2.17}$$

Since the matrix $\mathbf{A}^{t}$ transforms the vector **r** into the vector **r'**, it can be considered as an operator specifying a transformation of vectors. Then one can write by convention:

$$r' = \mathbf{A}^{\mathrm{T}}r. \tag{2.18}$$

The obtained conventional equality (2.18) is a corollary of the equality (2.17), which specifies the components of the new vector in matrix form. This is another form for specifying motion, with which the rotation of a vector is specified relative to a fixed coordinate system.

It should be noted that the transformations (2.17) and (2.18) are a form of the general affine transformation (linear transformation; cf., for example, [28]). The matrix relation (2.17) defines the following orthogonal transformation:

$$r'_j = \sum_{i=1}^{3} a_{ij} r_i. \tag{2.19}$$

It is important to note that the relations (2.17) or (2.18) are formally similar to the relations (2.15) or (2.3). Consequently, the latter are often treated also as an operation transforming a vector into a vector. However, the formulae (2.3) and (2.15) define the coordinates of the same vector in different bases, whereas the relations (2.17) and (2.18) actually define different vectors in one basis (in our case, in the basis **I**). Because of this, the transformation of **R** into **R'** is accomplished by the matrix **A** in the case of (2.15), and by the inverse matrix $\mathbf{A}^{t}$ in the case of (2.17).

The operation (2.18) transforming a vector into a vector permits one to specify the transformation of bases in yet another form:

$$\boldsymbol{e}_k = \mathbf{A}^T \boldsymbol{i}_k \quad (k = 1, 2, 3), \tag{2.20}$$

by which each basis vector of the basis **I** is transformed into a basis vector of the basis **E** by using the transformation (2.18). /50 One can verify that the basis vectors $\mathbf{e}_k$ form an orthogonal system of axes. Then it should be noted that in the first case considered of basis transformation (2.2) or (2.13) the basis vectors of the new basis **E** are expressed through the basis vectors of the old basis **I**, whereas in the transformation (2.20) the components of the basis vectors of the new basis are expressed by the components of the basis vectors of the old basis, the components being taken in the same coordinate system.

In this connection, it is necessary to distinguish the following possible methods for specifying rigid body rotation:

a) The transformation is specified in the form of the basis transformation operation (2.2), (2.13) or, which is exactly the same, in the form of a transformation of the components of an invariant vector with a basis transformation (2.3), (2.15). The vector rotation is then specified by the inverse operation;

b) The transformation is specified in the form of the vector rotation operation (2.18) or (2.17). If a basis rotation also accompanies this, then the position of the transformed basis relative to the reference is determined by the inverse operation.

It is easy to understand the reason for the difference of these two interpretations: for the same rotation of the body coordinate system relative to the reference, the motion of a

body vector in the fixed coordinate system appears opposite to the motion of a fixed vector relative to the body coordinate system. Although the orthogonal transformations (2.15) and (2.17) are formally identical, it is necessary to indicate, in addition, the form of the operation being specified*.

The matrix of the orthogonal transformation **A** has a determinant equaling +1. Furthermore, the transformation defined by the matrix **A** leaves one axis fixed. The well-known Euler theorem is a consequence of this, which states that any motion of a rigid body having a fixed point is equivalent to a rotation about a fixed axis. The axis of rotation is the eigenvector of the matrix **A** corresponding to the eigenvalue +1, and is determined from the characteristic equation **:

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$$(\mathbf{A} - \lambda 1)\,\mathbf{R} = 0,$$

where λ is the eigenvalue of the matrix.

The characteristic equation in the coordinates has the following form:

$$\left.\begin{aligned} (a_{11}-\lambda)r_1 + a_{12}r_2 + a_{13}r_3 &= 0,\\ a_{21}r_1 + (a_{22}-\lambda)r_2 + a_{23}r_3 &= 0,\\ a_{31}r_1 + a_{32}r_2 + (a_{33}-\lambda)r_3 &= 0, \end{aligned}\right\} \qquad (2.21)$$

and its solution for $\lambda = 1$ determines the vector of the axis of rotation to within an arbitrary multiple. For the solution of this equation, it is necessary to examine how the operator changes with a transformation of the basis.

Let the vector **s** be transformed by the operator **G** into the vector **t** [analogous to (2.18)]:

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* This question is discussed by Goldstein [16].

** The proof of the theorem is discussed in Goldstein's book, and we consider it unnecessary to repeat it here.

$$t = Gs, \tag{2.22}$$

and we have for the column matrices **T** and **S** analogous to (2.17):

$$\mathbf{T} = \mathbf{GS}; \tag{2.23}$$

The operator **G** is specified by the matrix $\|g_{ik}\|$.

Further, let there be a transformation of the coordinate system implemented by the matrix **A**. Then the column matrices defining the vectors **t** and **s** in the new basis will be found in accordance with (2.15) as:

$$\mathbf{T}' = \mathbf{AT}, \quad \mathbf{S}' = \mathbf{AS}. \tag{2.24}$$

On the other hand, the matrix equality (2.23) reflecting the vector equality (2.22) must also be subject to the transformation (2.15), i.e., we have:

$$\mathbf{AT} = \mathbf{AGS}. \tag{2.25}$$

Hence, one can obtain:

$$\mathbf{T}' = \mathbf{AT} = \mathbf{AGS} = \mathbf{AGA}^{-1}\mathbf{AS} = \mathbf{AGA}^{-1}\mathbf{S}' = \mathbf{G}'\mathbf{S}', \tag{2.26}$$

i.e., the equality (2.26) analogous to the equality (2.23) responds to the vector equality (2.22) in the new basis, the matrix of the operator **G** also being transformed to the new coordinates. Such a transformation is called a similar transformation and has the following form in accordance with (2.26): / 52

$$\mathbf{G}' = \mathbf{AGA}^{-1} = \mathbf{AGA}^{\mathrm{T}}. \tag{2.27}$$

In general, the characteristic Equation (2.21) has three roots. By denoting the number of the root by a second index, one can write the characteristic Equation (2.21) in the following form:

$$\sum_{j=1}^{3} a_{ij} r_{jk} = \lambda_k r_{ik} = \sum_{j=1}^{3} r_{ij} \delta_{jk} \lambda_k.$$

If one denotes the matrix $\|r_{jk}\| = \mathbf{R}$, then the obtained characteristic equation can be written in the matrix form:

$$\mathbf{AR} = \mathbf{R}\Lambda,$$

where:

$$\Lambda = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix}$$

is a diagonal matrix containing the eigenvalues. It follows from the obtained relation that:

$$\Lambda = \mathbf{R}^{-1}\mathbf{AR},$$

i.e., in order to find the eigenvalues of the matrix **A**, it is necessary to find the basis transformation for which the matrix **A** is reduced to diagonal form. It is known that the orthogonal transformation matrix has one eigenvalue equaling +1 (and two complex conjugates (16)), for which the eigenvector corresponds to the Euler rotation axis.

Let the matrix **B** give the orthogonal transformation to the axes of the basis **E'** in which the transformation of the matrix **A** is a rotation about the axis e'_1, i.e., the matrix A' determined in accordance with (2.27) as:

$$\mathbf{A}' = \mathbf{BAB}^{\mathrm{T}}, \tag{2.28}$$

has the form: /53

$$\mathbf{A}' = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & \sin\delta \\ 0 & -\sin\delta & \cos\delta \end{vmatrix}, \tag{2.29}$$

where δ is the angle of rotation about the rotational axis.

Obviously, the position of the natural rotational axis of the matrix A — i.e., the direction e'_1 — is determined by the first row of the matrix **B** [in accordance with (2.15)], whose

elements must also be solutions of the characteristic Equation (2.21). In accordance with (2.29), we can write:

$$\mathbf{A}=\mathbf{B}^{\mathrm{T}}\mathbf{A}'\mathbf{B}.$$

By calculating the obtained product of the three matrices, we generate the differences of the elements of the matrix **A**, which are symmetric with respect to its diagonal:

$$\left.\begin{aligned} a_{12}-a_{21}&=2b_{13}\sin\theta,\\ a_{31}-a_{13}&=2b_{12}\sin\theta,\\ a_{23}-a_{32}&=2b_{11}\sin\theta. \end{aligned}\right\} \tag{2.30}$$

The trace of the matrix **A** is not changed by similar transformations and equals:

$$\mathrm{Sp}\,\mathbf{A}=a_{11}+a_{22}+a_{33}=\mathrm{Sp}\,\mathbf{A}'=1+2\cos\theta. \tag{2.31}$$

The elements b_{ij}, which are the direction cosines of the eigenvector (Euler rotation axis) of the matrix **A**, are found from the obtained equalities (2.30) and (2.31):

$$\left.\begin{aligned} r_1=b_{11}&=(a_{23}-a_{32})\left[4-(1-a_{11}-a_{22}-a_{3.})\right]^{-\frac{1}{2}},\\ r_2=b_{12}&=(a_{31}-a_{13})\left[4-(1-a_{11}-a_{22}-a_{33})\right]^{-\frac{1}{2}},\\ r_3=b_{13}&=(a_{12}-a_{21})\left[4-(1-a_{11}-a_{22}-a_{23})\right]^{-\frac{1}{2}}. \end{aligned}\right\} \tag{2.32}$$

The basic relations considered here, which define orthogonal transformations, are sufficient for the subsequent investigations. The reader can find a more detailed examination of this problem in the literature cited.

§ 2.2. REPRESENTATION OF ORTHOGONAL TRANSFORMATIONS IN THE FORM OF QUATERNION PRODUCTS

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The representation of an orthogonal transformation defined by the matrix of direction cosines **A** discussed in the preceding paragraph is the most general method for specifying rigid body motion. The matrix **A** itself contains three independent parameters. Two of them determine the orientations of the axis of

the Euler plane rotation vector (i.e., the quantities (2.32) r_1, r_2, r_3 under the conditions $r_1^2+r_2^2+r_3^2=1$ and one — the angle of rotation ϑ.

Accordingly, the elements $\|a_{ij}\|$ can be expressed as functions of these three parameters. However, the matrix form for describing the orthogonal transformations then becomes very inconvenient and dependent on the representation of the operation in some basis or other. The representation of orthogonal transformations with the use of quaternion algebra is significantly more convenient. This representation is based on the fundamental theorem 1.1 considered in paragraph 1.3. In accordance with this theorem, the rotation operation (1.14) specifies the transformation of the vector part of the quaternion **r** into **r'** (1.20) and thus can be considered as an orthogonal transformation in real three-dimensional space.

Let the vector:

$$r=\sum_{i=1}^{3} r_i i_i.$$

be specified in a three-dimensional basis I generated by the basis vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$. Further, let the matrix **B** define the linear orthogonal transformation of the vector **r** into the vector **r'** specified in the form (2.19) which can also be written in operator form (2.18):

$$r'=Br$$

(in this case $\mathbf{B}=\mathbf{A}^T$).

We will consider the four-dimensional hypercomplex space H generated by the hypercomplex units i_1, i_2, i_3 which formally coincide with the basis vectors of the three-dimensional basis I.

Let the rotation operation (1.20) determined by the unit quaternion be specified in the space **H** which puts the vector **r** into correspondence with the vector **r'** / 55

$$r' = \Lambda \circ r \circ \tilde{\Lambda}. \tag{2.33}$$

Obviously, the vectors **r'** and **r** are defined by their components in the hypercomplex basis **H**. The rotation operation (1.20), by analogy to the operation (2.20), then specifies the transformation of a vector into a vector, which is accomplished relative to the fixed basis. We write these two transformations in coordinate form obtained from formulae (2.19) and (1.19), respectively:

$$r'_i = \sum_{j=1}^{3} b_{ij} r_j, \tag{2.34}$$

$$\left.\begin{aligned} r'_1 &= (\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2) r_1 + 2(\lambda_1\lambda_2 - \lambda_0\lambda_3) r_2 + \\ &\quad + 2(\lambda_1\lambda_3 + \lambda_0\lambda_2) r_3, \\ r'_2 &= 2(\lambda_1\lambda_2 + \lambda_0\lambda_3) r_1 + (\lambda_0^2 + \lambda_2^2 - \lambda_1^2 - \lambda_3^2) r_2 + \\ &\quad + 2(\lambda_2\lambda_3 - \lambda_0\lambda_1) r_3, \\ r'_3 &= 2(\lambda_1\lambda_3 - \lambda_0\lambda_2) r_1 + 2(\lambda_2\lambda_3 + \lambda_0\lambda_1) r_2 + \\ &\quad + (\lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2) r_3. \end{aligned}\right\} \tag{2.35}$$

We identify the coefficients b_{ij} of the transformation (2.34) with the coefficients of the linear transformation (2.35). A unique correspondence between the forms for representing an orthogonal transformation (1.20) and (2.18) is then established. This correspondence is isomorphic, i.e., any operation of a group of rotations in the three-dimensional space **I** defined by the orthogonal matric **B** (the operator **B**) corresponds to the rotation operation in the hypercomplex space specified by the quaternion **Λ**, **B** and **Λ** being mutually defined by each other.

Actually, we will consider the conditions of the group.

1. To the inverse operation (2.34) or (2.18)

$$r = \mathbf{B}^{-1} r'$$

there corresponds the transformation:

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$$r = \tilde{\Lambda} \circ r' \circ \Lambda,$$

Then the equalities $\mathbf{B}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{B} = 1$ and $\Lambda \circ \tilde{\Lambda} = \tilde{\Lambda} \circ \Lambda = 1$ hold, i.e., there is an identity defined by the identical transformation.

2. To the product of two transformations:

$$r' = \mathbf{B} r, \quad r'' = \mathbf{C} r',$$

i.e., the transformation defined by the matrix operator D

$$r'' = \mathbf{D} r, \quad \mathbf{D} = \mathbf{C}\mathbf{B},$$

there corresponds the product of the quaternions of the successive transformations:

$$r' = \Lambda \circ r \circ \tilde{\Lambda}, \quad r'' = \mathbf{M} \circ r' \circ \tilde{\mathbf{M}}, \quad r'' = \mathbf{N} \circ r \circ \tilde{\mathbf{N}},$$

where:

$$\mathbf{N} = \mathbf{M} \circ \Lambda.$$

The quaternion components can serve as kinematic parameters by analogy to the elements of the matrix **B**, which are the direction cosines, since they uniquely define the position of one basis with respect to another. Hence, it follows that they can be used for describing rigid body motion relative to some selected coordinate system. It should then be kept in mind that two quaternions Λ and $-\Lambda$ defining the same rotation transformation correspond to one position of the body.

It follows directly from the formulae (2.34) and (2.35) that the components of the quaternion Λ define the elements of the matrix **B**. Hence, one can also determine the components of Λ (as should be expected, they are obtained with two signs), if the quantities b_{ij} are given. Actually, in accordance with (2.35) we have

$$b_{11}+b_{22}+b_{33}=3\lambda_0^2-\lambda_1^2-\lambda_2^2-\lambda_3^2=4\lambda_0^2-1, \qquad (2.36)$$

$$\lambda_0=\pm\frac{\sqrt{1+b_{11}+b_{22}+b_{33}}}{2}$$

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and, by considering the difference of the elements symmetric with respect to the diagonal, we obtain:

$$\left.\begin{aligned}\lambda_1&=\pm\frac{b_{32}-b_{23}}{4\lambda_0}=\frac{b_{32}-b_{23}}{2\sqrt{1+b_{11}+b_{22}+b_{33}}},\\ \lambda_2&=\pm\frac{b_{13}-b_{31}}{4\lambda_0}=\frac{b_{13}-b_{31}}{2\sqrt{1+b_{11}+b_{22}+b_{33}}},\\ \lambda_3&=\pm\frac{b_{21}-b_{12}}{4\lambda_0}=\frac{b_{21}-b_{12}}{2\sqrt{1+b_{11}+b_{22}+b_{33}}}.\end{aligned}\right\} \qquad (2.37)$$

By comparing the relations (2.36), (2.37) and (2.31), (2.32), one can easily verify that the vector part of the quaternion Λ is the axis of the Euler rotation, i.e., it is the eigenvector of the matrix $\mathbf{B}$ (one should take the fact into account that in our case $\mathbf{B}=\mathbf{A}^t$ and the components λ_i are the direction cosines of the unit vector ζ of the quaternion Λ).

§ 2.3. BASIS TRANSFORMATION

The correspondence obtained between the two forms for representing the transformation of a vector $\mathbf{r}$ into the vector $\mathbf{r}'$ permits the orthogonal transformation operation (2.19) to be written in the form (1.20). The transformation of (1.20) will always transform a vector, since the basis I remains fixed, formally coinciding with the basis vectors $\mathbf{i}_j$ of the hypercomplex space H in which the quaternion multiplication operation is accomplished. Nevertheless, one can find, with the known operations for transforming bases and coordinates of a fixed vector, the quaternion operation uniquely corresponding to them.

We will consider the problem of basis transformation. Let there be given in three-dimensional space, in addition to the basis I, the orthonormal basis E with the basis vectors $\mathbf{e}_i$,

where:

$$e_k = \sum_{i=1}^{3} e_{ki} i_i \qquad (i,\ k = 1,\ 2,\ 3). \tag{2.38}$$

Let us further consider the orthogonal transformation occurring on the components of the vectors $\mathbf{e}_k$:

$$e'_{ki} = \sum_{j=1}^{3} b_{ij} e_{kj}. \tag{2.39}$$

Obviously, the elements e'_{ki} define new vectors: /58

$$e'_k = \sum_{i=1}^{3} e'_{ki} i_i.$$

which are obtained from $\mathbf{e}_k$ by the orthogonal transformation (2.39). This transformation can also be written in matrix form:

$$e'_k = \mathbf{B} e_k \qquad (k = 1,\ 2,\ 3). \tag{2.40}$$

One can verify that the vectors $\mathbf{e'}_k$ form an orthonormal basis $\mathbf{E'}$ (this is satisfied because of the orthogonality conditions of the matrix $\mathbf{B}$). In view of the isomorphism of the orthogonal transformation operation (2.34) and the rotation operation (1.20), the transformation of bases (2.39) or (2.40) can be written in the form:

$$e'_k = \Lambda \circ e_k \circ \tilde{\Lambda} \qquad (k = 1,\ 2,\ 3). \tag{2.41}$$

Such a transformation actually defines the basis transformation. The vectors $\mathbf{e'}_k$ will be unit vectors (since the rotation transformation does not change the vector norm) and mutually orthogonal. By considering the scalar product, one can see that the orthogonality condition is satisfied:

$$e'_k \cdot e'_n = \frac{1}{2}(e'_k \circ e'_n + e'_n \circ e'_k) =$$
$$= \frac{1}{2}(\Lambda \circ e_k \circ e_n \circ \tilde{\Lambda} + \Lambda \circ e_n \circ e_k \circ \tilde{\Lambda}) = \frac{1}{2}\Lambda \circ (e_k \cdot e_n) \circ \tilde{\Lambda} = \delta_{kn},$$

i.e., the vectors $\mathbf{e'}_k$ form the basis $\mathbf{E'}$.

Hence, it is seen that the basis transformation specified in the form of the rotation operation (2.41) corresponds to only one

of the forms considered above for specifying rigid body motion, namely, to the orthogonal transformation in the form (2.20).

Another form for representing the basis transformation (2.2) and (2.13) expresses the basis vectors of the new basis in the form of their resolution along the vectors of the reference basis. Such a transformation form cannot be written in the form of a rotation operation (1.20). Nevertheless, the elements of the matrix of the transformations (2.2) and (2.3) can be expressed by the matrix elements of the rotation transformation, / 59 since the transformations in the form (2.18), (2.40) and in the form (2.2), (2.13) have mutually inverse matrixes (i.e., transposed matrices).

Actually, let the basis transformation be specified in the form (2.39). We will show that the basis vectors of the new basis $\mathbf{e}'_k$ are expressed by the basis vectors of the reference basis $\mathbf{e}_k$ with the help of the inverse matrix B^T. We have, in accordance with (2.39):

$$e'_k = \sum_{i=1}^{3} e'_{ki} i_i = \sum_{i,j=1}^{3} b_{ij} e_{kj} i_i = \sum_{i,j,m=1}^{3} b_{ij} e_{kj} e_{mi} e_m.$$

where the basis vectors $\mathbf{i}_i$ are expressed with the help of the transformation, the inverse of (2.38). Since the elements $\|e_{kj}\|$ form an orthogonal transformation basis, then the following equality is valid

$$\sum_{k,m=1}^{3} e_{km} e_{mk} = 1.$$

By substituting this unit into the sum determining $\mathbf{e}'_k$, we obtain:

$$e'_k = \sum_{i,j,m,k,m=1}^{3} b_{ij} e_{kj} e_{km} e_{mk} e_{mi} e_m = \sum_{i,j,m=1}^{3} b_{ij} \delta_{jm} \delta_{ki} e_m =$$
$$= \sum_{m=1}^{3} \left(\sum_{j,i=1}^{3} b_{ji} \delta_{jm} \delta_{ki} \right) e_m = \sum_{m=1}^{3} b_{mk} e_m \qquad (2.42)$$

Here the orthogonality conditions (2.4) were used. It follows from (2.42) that the elements of the matrix $\mathbf{B}^T$, which determine the resolution of the vectors of the basis $\mathbf{E}'$ along the vectors of the basis $\mathbf{E}$, are equivalent to the elements of the rotation transformation matrix (2.35) specified by the inverse quaternion $\tilde{\Lambda}$.

Let us further consider the representation of the vectors of the basis $\mathbf{E}$ (2.38). It can be seen that the coefficients e_{ki} form an orthogonal transformation matrix changing the basis vectors $\mathbf{i}_k$ into the basis vectors $\mathbf{e}_k$. The form of the transformation, by analogy to (2.39), will be the obvious identity:

$$e_{ki} = \sum_{j=1}^{3} e_{ji}\delta_{kj}, \tag{2.43}$$

which makes the new components e_{ki} of the basis $\mathbf{E}$ agree with the unit components of the resolution of the reference bases (i.e., $\mathbf{I}$). In view of the isomorphism of orthogonal transformation operation and the rotation operation, there exists a quaternion S, whose components are determined by the elements e_{kj} from relations analogous to (2.36) and (2.37), which satisfies the equality: /60

$$\mathbf{e}_k = S \circ \mathbf{i}_k \circ \tilde{S}. \tag{2.44}$$

One can see from this representation that the elements of the inverse (transposed) matrices $\|e_{ki}\|$ take part in the formula of the vector resolution (2.38) and the orthogonal transformation (2.43).

We note once again that the transformation (2.44) specifies the rotation of a vector, i.e., it is isomorphic to the transformation (2.40) and not to (2.2).

§ 2.4. TRANSFORMATION OF COMPONENTS OF A FIXED VECTOR HYPERCOMPLEX MAPPING

In the preceding two paragraphs, the vector transformation and the basis transformation were specified in the form of the rotation operation (1.20). Let us now consider how the transformation of the components of a fixed vector, which occurs due to a change of basis, can be represented in the same form.

Let us consider the fixed vector **r**. Let its projection on the axes of the bases **E** and **E'** be the quantities r_{iE} and $r_{iE'}$ respectively:

$$r = \sum_{i=1}^{3} r_{iE} e_i = \sum_{i=1}^{3} r_{iE'} e'_i. \tag{2.45}$$

We introduce the following quaternions composed of the components of the vector in the bases **E** and **E'**:

$$\left.\begin{aligned} R_E &= r_{1E} i_1 + r_{2E} i_2 + r_{3E} i_3, \\ R_{E'} &= r_{1E'} i_1 + r_{2E'} i_2 + r_{3E'} i_3. \end{aligned}\right\} \tag{2.46}$$

We will call the quaternions (2.46) hypercomplex mappings (or simply mappings) of the vector **r** onto the bases **E** and **E'** respectively. The hypercomplex mapping of a vector onto some basis coincides with the vector itself, if this basis coincides with the three vectors i_1, i_2, i_3 of the hypercomplex space **H** or coincides with the basis **I**, which is the same thing. / 61

The basis vectors of the basis **E'** are expressed by the basis vectors of the basis **E** with the help of the rotation operation (2.41). We substitute the relations (2.41) into the equalities (2.45):

$$r = \sum_{i=1}^{3} r_{iE'} e'_i = \sum_{i=1}^{3} r_{iE'} \Lambda \circ e_i \circ \tilde{\Lambda} = \Lambda \circ \left(\sum_{i=1}^{3} r_{iE'} e_i \right) \circ \tilde{\Lambda}.$$

If the basis vectors of the basis **E** coincide with the vectors i_1, i_2, i_3 of the hypercomplex space, then in the obtained equality:

$$r = \sum_{i=1}^{3} r_{iE} e_i = \Lambda \circ \left(\sum_{i=1}^{3} r_{iE'} e_i \right) \circ \tilde{\Lambda}$$

the quantities $\sum_{i=1}^{3} r_{iE} e_i$ and $\sum_{i=1}^{3} r_{iE'} e_i$ coincide with the hypercomplex mappings of the vector **r** onto the bases **E** and **E'** (2.46). Hence it follows that:

$$R_E = \Lambda \circ R_{E'} \circ \tilde{\Lambda}.$$

By multiplying this expression on the left by $\widetilde{\Lambda}$ and on the right by Λ, we obtain:

$$R_{E'} = \tilde{\Lambda} \circ R_E \circ \Lambda. \qquad (2.47)$$

Thus, the following theorem is valid.

Theorem 2.1. If the transformation of the unit vectors of the basis **E** into the vectors of the basis **E'** is defined by the rotation operation (2.41), then the transformation of the coordinates of a fixed vector onto these bases is specified by the inverse operation (2.47) performed on the mappings.

The obtained result is natural, since orthogonal transformations specifying a basis transformation in the form (2.20) and the transformation of the components of the fixed vector (2.15) are inverse. Since the rotation operations (2.33) and (2.41) specify the transformations of unit vectors, each of which is represented in quaternion form, i.e., specified by their components in the basis **I**, then the inverse operation (2.47), which specifies /62
the transformation of the components of a fixed vector in these bases, is performed on the quaternions which are also the mappings (2.46). The components $r_{iE'}$ of the vector **r** can be found from the components $r_{iE'}$, if the quaternion multiplication

is performed in formula (2.47) and the elements of the basis vectors of the space **H** are equated.

In accordance with theorem 2.1, one can expect that if for a basis transformation (for example, with the transformation (2.41)) certain quantities are transformed by the inverse operation (for example, (2.47)), then they define a fixed vector. We will find the relation between the projections of the vector **r** onto the basis **E** and its projections on the basis **I**. We denote the projections of the vector **r** onto the basis **I** by the quantities r_{iI}:

$$\cdot = \sum_{i=1}^{3} r_{iI} i_r$$

The transition from the basis **I** to the basis **E** is determined by the quaternion **S** with the help of the rotation operation (2.44). In accordance with theorem 2.1, the components of r_{iI} must be expressed by r_{iE} with the help of the inverse operation performed on the mappings. Actually, the mapping of **r** onto the basis **I** coincides with the vector itself:

$$R_I = r = \sum_{i=1}^{3} r_{iI} i_I = \sum_{i=1}^{3} r_{iE} e_i.$$

By replacing e_i by its value (2.44) we obtain:

$$R_I = \sum_{i=1}^{3} r_{iE} S \circ i_i \circ \tilde{S} = S \circ \left(\sum_{i=1}^{3} r_{iE} i_i \right) \circ \tilde{S} = S \circ R_E \circ \tilde{S},$$

from which it follows:

$$R_E = \tilde{S} \circ R_I \circ S. \qquad (2.48)$$

§ 2.5. VARIANCE OF THE ROTATION OPERATION. RODRIGUEZ - HAMILTON PARAMETERS

Now, let us consider how the operation (2.33) for the transformation of the vector **r** into the vector **r'** appears, if it is written in projections of the vector **r**, not onto the basis I, but onto another basis, say E. We dencte the projections of the vector **r** onto the bases E by $\mathbf{r}_{iE}$, and the projections of the vector **r'** onto the basis E by $\mathbf{r'}_{iE}$. We have:

$$r = \sum_{i=1}^{3} r_{iE} e_i = \sum_{j=1}^{3} r_{jI} i_j = R_I,$$

$$r' = \sum_{i=1}^{3} r'_{iE} e_i = \sum_{j=1}^{3} r'_{jI} i_j = R'_I,$$

where r_{jI}, r'_{jI} are the projections of the vectors **r** and **r'** on the basis I (denoted earlier as r_i) and r'_i and R_I and R'_I are the hypercomplex mappings of the vectors **r** and **r'** on the basis I coinciding with the vectors **r** and r' onto the basis E: / 63

$$R_E = \sum_{i=1}^{3} r_{iE} i_i, \quad R'_E = \sum_{i=1}^{3} r'_{iE} i_i.$$

In accordance with theorem 2.1 the projections of the vectors **r** and **r'** onto the basis vectors of the basis E are coupled with their projections onto the basis vectors of the basis I by the operation performed upon the hypercomplex mappings and the inverse basis transformation operation. Since the transformation of the basis vectors of the basis E is determined by the operation (2.44), then by analogy to (2.48) we have:

$$R_E = \tilde{S} \circ R_I \circ S, \quad R'_E = \tilde{S} \circ R'_I \circ S.$$

Let us find the relation of R'_E and R_E. The mappings of R'_I and R_I coincide with the vectors $\mathbf{r}'$ and $\mathbf{r}$ and consequently they are related by the relations (2.33). Thus:

$$R'_E = \tilde{S} \circ R'_I \circ S = \tilde{S} \circ \Lambda \circ R_I \circ \tilde{\Lambda} \circ S = \\ = \tilde{S} \circ \Lambda \circ S \circ R_E \circ \tilde{S} \circ \tilde{\Lambda} \circ S = \Lambda_E \circ R_E \circ \tilde{\Lambda}_E, \qquad (2.49)$$

where the quaternion $\tilde{S} \circ \Lambda \circ S$ is denoted by Λ_E, and then, obviously, $\tilde{\Lambda}_E = \tilde{S} \circ \tilde{\Lambda} \circ S$. One can see (cf. paragraph 1.3) that the quaternion Λ_E has the same scalar part as the quaternion Λ. The vector part of the quaternion Λ_E is obtained from the vector part of the quaternion Λ by the rotation transformation (1.20), i.e.:

$$\lambda_E = \tilde{S} \circ \lambda \circ S.$$

One can conclude from this equality by using the results of theorem 2.1 that the vector part of the quaternion Λ specifying the rotation operation (2.33) is a fixed vector, since the basis transformation is changed in accordance with the rules of the transformation of the coordinates of a fixed vector. In addition, it is easily noted that the transformation operation (2.45) maintained its form identically with (2.33). Hence, we have the following theorem: 64

Theorem 2.2. The rotation operation (2.33) is invariant relative to a basis transformation if the vectors $\mathbf{r}$ and $\mathbf{r}'$ and the vector part of the quaternion of the transformation Λ are expressed in the coordinates of the same arbitrary basis; the transformation of the coordinates of these vectors is carried out in accordance with the rules for transforming an invariant vector. Thus, the operation for transforming the vector $\mathbf{r}$ into the vector $\mathbf{r}'$ can be written in the components of $\mathbf{r}$ and $\mathbf{r}'$ in any basis. The vector part of the quaternion Λ should then also be projected onto the same basis.

In view of the fact that one can write:

$$r = \sum_{i=1}^{3} r_{iE} e_i, \quad r' = \sum_{i=1}^{3} r'_{iE} e_i, \quad \Lambda = \lambda_0 + \sum_{i=1}^{3} \lambda_{iE} e_i, \tag{2.50}$$

The content of theorem 2.2 can also be interpreted so that the operation (2.49) can be described in the form (1.20) on the quantities (2.50). Then the "rules" for multiplying the basis vectors e_i are analogous to the rules for multiplying the hypercomplex numbers. It is easily understood that this is due to the fact that the basis vectors i_1, i_2, i_3 of the hypercomplex space **H** can be made to coincide with any basis of real three-dimensional space. Then the vectors of the three-dimensional space coincide with the quaternions, the mappings of these vectors.

Theorem 2.2 establishes yet another peculiarity of the rotation operation (2.33) — the fact that this operation has a vector meaning. By using the representation of quaternion multiplication with the help of the vector multiplication operation (1.9), one can represent the transformation (2.33) with the help of the vector operations in three-dimensional space.

It is necessary for this to represent the transformation quaternion Λ as a function of a three-dimensional vector. This can be done by considering the quantity: /65

$$\frac{\Lambda}{\lambda_0} = 1 + \frac{\lambda_1}{\lambda_0} i_1 + \frac{\lambda_2}{\lambda_0} i_2 + \frac{\lambda_3}{\lambda_0} i_3 = 1 + \frac{\lambda}{\lambda_0} = 1 + \frac{1}{2}\theta,$$

which determines the quaternion Λ by the vector:

$$\theta = 2\frac{\lambda}{\lambda_0}. \tag{2.51}$$

The vector θ is called the finite rotation vector [31, 44]. By using the representation of the quaternion in the form (1.11), one can see that the finite rotation vector equals:

$$\theta = 2\zeta \operatorname{tg}\frac{\Phi}{2}. \tag{2.52}$$

By using formula (1.9), we replace the quaternion multiplication operation in relation (2 33) by the vector multiplication operation in three-dimensional space:

$$\begin{aligned} r' = \Lambda \circ r \circ \tilde{\Lambda} &= (\lambda_0 + \lambda) \circ r \circ (\lambda_0 - \lambda) = \\ &= (\lambda_0 r - \lambda \cdot r + \lambda \times r) \circ (\lambda_0 - \lambda) = \\ = \lambda_0^2 r - \lambda_0 \lambda \cdot r + \lambda_0 \lambda \times r + \lambda_0 r \cdot \lambda - \lambda_0 r \times \lambda + (\lambda \cdot r)\lambda + \\ + (\lambda \times r) \cdot \lambda - (\lambda \times r) \times \lambda. \end{aligned}$$

By reducing the similar terms and taking into account that $(\lambda \times r) \cdot \lambda = 0$, and also that $(\lambda \times r) \times \lambda = \lambda \times (r \times \lambda) = r(\lambda \cdot \lambda) - \lambda(r \cdot \lambda)$, we write the obtained expression for r' in the form:

$$\begin{aligned} r' &= \lambda_0^2 r + 2\lambda_0 \lambda \times r + r\lambda^2 - 2(r\lambda^2 - \lambda r \cdot \lambda) = \\ &= r + 2\lambda_0 \lambda \times r - 2\lambda \times (r \times \lambda) = \\ &= r + \lambda_0^2 \frac{2\lambda}{\lambda_0} \times \left(r - \frac{1}{2} r \times \frac{2\lambda}{\lambda_0} \right). \end{aligned}$$

where:

$$\lambda^2 = \lambda \cdot \lambda = \lambda_1^2 + \lambda_2^2 + \lambda_3^2.$$

We use the finite rotation vector introduced in (2.51) by considering that, in accordance with (2.52),

$$\lambda_0^2 = \cos^2 \frac{\Phi}{2} = \frac{1}{1 + \frac{1}{4}|\theta|^2}.$$

We obtain the well-known Rodriguez formula for transforming the vector with finite rotation:

$$r' = r + \frac{1}{1+\frac{1}{4}|\theta|^2}\theta \times \left(r + \frac{1}{2}\theta \times r\right). \tag{2.53}$$

We further note that the quaternion Λ, which specifies the operation (2.41) of the orthogonal transformation of the basis **E** into basis **E'**, also determines the finite rotation vector θ (2.51). This transformation is actually a rotation about the axis θ, since the vector θ itself with such a transformation is not changed:

$$\Lambda \circ \theta \circ \tilde{\Lambda} = \frac{2}{\lambda_0}\Lambda \circ \lambda \circ \tilde{\Lambda} = \frac{2}{\lambda_0}\lambda = \theta.$$

By using a representation of a quaternion in the form (1.11), one can write its components in the following form:

$$\lambda_0 = \cos\frac{\Phi}{2}, \quad \lambda_1 = \gamma_1 \sin\frac{\Phi}{2}, \quad \lambda_2 = \gamma_2 \sin\frac{\Phi}{2}, \quad \lambda_3 = \gamma_3 \sin\frac{\Phi}{2}. \tag{2.54}$$

Where $\gamma_1, \gamma_2, \gamma_3$ are the direction cosines of the vector θ, as well as λ, in the axes of the basis under consideration. Correspondingly, the finite rotation vector or the transformation quaternion can be specified by its components in any basis. By analogy to the hypercomplex mappings of a fixed vector onto some basis, we will consider the quaternion components composed of the components (2.54), where γ_i is the projection of the vector ζ of the quaternion on a given basis. The latter is called the mapping of the quaternion Λ onto the basis under consideration. Obviously, the quaternion mapping has the same scalar part as the initial quaternion and the vector part equaling the mapping of the vector part of the quaternion onto this basis. Thus, the mapping of the quaternion $\Lambda = \lambda_0 + \lambda$ onto the basis **E** will be the quaternion introduced previously (cf. 2.49):

$$\Lambda_E = \lambda_0 + \lambda_E = \lambda_0 + \sum_{i=1}^{3} \lambda_{iE} i_i = \tilde{S} \circ \Lambda \circ S.$$

We will denote the mapping of the quaternion onto the basis by the index of this basis in contrast to the initial quaternion without an index. In accordance with theorem 2.1, the mapping of a quaternion is found by the rotation operation, the inverse of the basis transformation. /67

The components of the quaternion in the coordinate system, which is generated by this quaternion, have a peculiar meaning. In the case under consideration, the basis **E** transformed by the quaternion **Λ** into the basis **E'** is such a system. The transformation of the basis **E** into the basis **E'** is determined by the operation (2.41). In accordance with theorem 2.2, this transformation can be written in the projections of the unit vectors e_i and e'_i and the quaternion **Λ** onto the basis **E** (the same as onto any other basis).

The components of the quaternion **Λ** in the basis **E** are determined by the mapping Λ_E. In view of the fact that the transformation (2.41) of the basis vectors of the basis **E** into the basis vectors of the basis **E'**, in accordance with theorem 2.2, can be written in the form (2.49), i.e., in the form:

$$(e'_k)_E = \Lambda_E \circ (e_k)_E \circ \tilde{\Lambda}_E,$$

then the components of **Λ** in the basis **E'** determine the mapping of **Λ** onto the basis **E'** with the help of the inverse operation:

$$\Lambda_{E'} = \tilde{\Lambda}_E \circ \Lambda_E \circ \Lambda_E = \Lambda_E.$$

Hence, it is seen that the components of the quaternion **Λ** in the basis **E** and **E'** are identical. The quaternion components in the basis transformed by this quaternion, which are specified in the form (2.54), are called the Rodriguez-Hamilton parameters [31].

The quaternion whose components are the Rodriguez-Hamilton parameters will be noted below by an asterisk. This quaternion has equal components in two coordinate systems, because it is this quaternion which determines the transition from one coordinate system to another. In this connection, the quaternion whose parameters are the Rodriquez-Hamilton parameters has a peculiar meaning. It is as if it were an "eigen" quaternion of the transformation.

We will subsequently, by convention, call quaternions composed of the Rodriguez-Hamilton parameters eigenquaternions of the transformations. In this case:

$$\Lambda^* = \Lambda_E = \Lambda_{E'} = \tilde{S} \circ \Lambda \circ S.$$

will be such a quaternion. /68

Just as the transition from the basis **I** to the basis **E** determines the quaternion **S** (2.44), in accordance with theorem 2.1, we obtain the following magnitudes of the mappings of the quaternion **S** onto the bases **I** and **E**:

$$S_I = S, \quad S_E = \tilde{S} \circ S_I \circ S = S,$$

from which:

$$S^* = S_I = S_E = S.$$

§ 2.6. SUMMATION OF ROTATIONS

The addition of finite rotations occurs with the performance of a sequence of two, three, etc. orthogonal transformations. The multiplication operation of the matrices of the component transformations corresponds to this. If the first transition (for example, from the basis **E** to the basis **E'**) is specified by the matrix **A** and the second (from **E'** to **E"**) is determined by the matrix **B**, then the matrix of the resulting rotation **C** (from

the basis **E** to the basis **E"**) will be specified by the product:

$$\mathbf{C}=\mathbf{BA}. \tag{2.55}$$

If the transition operation from the basis **E** to the basis **E'** is determined by the isomorphic rotation operation specified by the quaternion **A** (2.41) and the transition operation from the basis **E'** to the basis **E"** is specified analogously by the quaternion **M"**:

$$e_i''=\mathbf{M}\circ e_i'\circ\tilde{\mathbf{M}} \qquad (i=1,\ 2,\ 3),$$

then the quaternion of the resulting transformation **N** defined by the transition operation:

$$e_i''=\mathbf{N}\circ e_i\circ\tilde{\mathbf{N}},$$

is found as the product of the quaternions of the transformation components:

$$\mathbf{N}=\mathbf{M}\circ\mathbf{A} \tag{2.56}$$

(this was considered in paragraph 1.3). We note here the similarity in the formulae of the operator multiplication (2.55) and (2.56).

The components of the quaternion $\mathbf{N}=\nu_0+\nu_1 i_1+\nu_2 i_2+\nu_3 i_3$ are determined by the components of the quaternions of the transformation components $\mathbf{A}=\lambda_0+\sum_{j=1}^{3}\lambda_j i_j$ and $\mathbf{M}=\mu_0+\sum_{j=1}^{3}\mu_j i_j$, in accordance with the rules of quaternion multiplication (1.5). By performing this multiplication and equating the coefficients of the basis vectors, we obtain four scalar equations: /69

$$\left.\begin{aligned}\nu_0&=\lambda_0\mu_0-\lambda_1\mu_1-\lambda_2\mu_2-\lambda_3\mu_3,\\ \nu_1&=\lambda_0\mu_1+\lambda_1\mu_0+\lambda_3\mu_2-\lambda_2\mu_3,\\ \nu_2&=\lambda_0\mu_2+\lambda_2\mu_0+\lambda_1\mu_3-\lambda_3\mu_1,\\ \nu_3&=\lambda_0\mu_3+\lambda_3\mu_0+\lambda_2\mu_1-\lambda_1\mu_2.\end{aligned}\right\} \tag{2.57}$$

The components λ_i, μ_i and ν_i of the component and resulting quaternion determine in the coordinate system **I** (i.e., in the

system where the quaternions are specified) the vectors of the first, second and resulting rotations. However, in the preceding paragraph it was shown that the rotation operation is valid in any coordinate system if the transformed vectors and the vector parts of the transformation quaternions are expressed in it. Let us consider how the transformation summation operation (2.56) appears if it is expressed in another basis.

Let the transition from the basis I to the basis E be determined by the quaternion S in accordance with (2.44). Then the quantities s_i $(i=0,1,2,3)$ will be the components of this quaternion in the bases I and E, since the mapping of the quaternion onto the basis I coincides with the quaternion itself $S_I = S$, and the mapping of S onto the basis E is determined by the rotation operation, the inverse of (2.44), i.e.:

$$S_E = \tilde{S} \circ S \circ S = S_I = S^*.$$

Let us consider the quaternion Λ determining the finite rotation converting the basis E into the basis E'. The quaternion:

$$\Lambda_E = \tilde{S} \circ \Lambda \circ S.$$

will be (cf. the preceding paragraph) the mapping of the quaternion Λ onto the basis E. Since the transition from the basis I to the basis E' is determined by the successive application of S and Λ, then, in agreement with (2.56), the resulting transition is determined by the product $\Lambda \circ S$. In accordance with this, the mapping of Λ onto the basis E' is found as: / 70

$$\Lambda_{E'} = \widetilde{(\Lambda \circ S)} \circ \Lambda \circ (\Lambda \circ S) = \tilde{S} \circ \tilde{\Lambda} \circ \Lambda \circ \Lambda \circ S = \\ = \tilde{S} \circ \Lambda \circ S = \Lambda_E = \Lambda^*, \qquad (2.58)$$

which was shown previously.

The transition from the basis I to the basis E" is determined by the sequence of transitions specified by the quaternions S, Λ, M. The resulting quaternion of the transition from the basis I to the basis E" will be equal to $M \circ \Lambda \circ S$. The quantity:

$$\Lambda_{E''} = \overline{(M \circ \Lambda \circ S)} \circ \Lambda \circ (M \circ \Lambda \circ S) = \tilde{S} \circ \tilde{\Lambda} \circ \tilde{M} \circ \Lambda \circ M \circ \Lambda \circ S. \qquad (2.59)$$

will be the mapping of the quaternion Λ onto the basis E".

By considering, by analogy, the quaternion M specifying the transition from the basis E' to the basis E", we find the magnitudes of the mappings M onto the bases E,E' and E":

$$\left.\begin{aligned} &M_E = \tilde{S} \circ M \circ S, \quad M_{E'} = \tilde{S} \circ \tilde{\Lambda} \circ M \circ \Lambda \circ S, \\ &M_{E''} = (\tilde{S} \circ \tilde{\Lambda} \circ \tilde{M}) \circ M \circ (M \circ \Lambda \circ S) = \\ &\qquad = \tilde{S} \circ \tilde{\Lambda} \circ M \circ \Lambda \circ S = M_{E'} = M^*. \end{aligned}\right\} \qquad (2.60)$$

The projections of the resulting rotation vector onto the bases E,E' and E" are determined by the mapping of the quaternion N onto these bases:

$$\left.\begin{aligned} &N_E = \tilde{S} \circ N \circ S, \quad N_{E'} = \tilde{S} \circ \tilde{\Lambda} \circ N \circ \Lambda \circ S, \\ &N_{E''} = \tilde{S} \circ \tilde{\Lambda} \circ \tilde{M} \circ N \circ M \circ \Lambda \circ S = \\ &\qquad = \tilde{S} \circ \tilde{N} \circ N \circ N \circ S = N_E = N^*. \end{aligned}\right\} \qquad (2.61)$$

We note that the mapping of the quaternion M onto the bases E' and E", as well as the mapping of N onto the bases E and E" are equal and determine the eigenquaternions (the Rodriguez-Hamilton parameters) of these transformations.

Theorem 2.3. The components of the resulting rotation quaternion (N) are determined by the components of the first (Λ) and second (M) rotations by the quaternion multiplication rule (2.56) and (2.57). The multiplication rule is preserved for mappings of these quaternions onto an arbitrary basis.

Proof: The formula (2.56) itself is a particular case of this thecrem, since the quaternions $\mathbf{N},\Lambda$ and $\mathbf{M}$ coincide with their mappings onto the basis $\mathbf{I}$. We will find the mapping of the quaternion $\mathbf{N}$ onto the basis $\mathbf{E}$ depending on the mappings of the component quaternions Λ and $\mathbf{M}$ onto this same basis. In accordance with the relations (2.56), (2.58)-(2.61), we have: /71

$$\mathbf{N}_E = \tilde{\mathbf{S}} \circ \mathbf{N} \circ \mathbf{S} = \tilde{\mathbf{S}} \circ \mathbf{M} \circ \Lambda \circ \mathbf{S} = \tilde{\mathbf{S}} \circ \mathbf{M} \circ \mathbf{S} \circ \tilde{\mathbf{S}} \circ \Lambda \circ \mathbf{S} = \mathbf{M}_E \circ \Lambda_E.$$

One can obtain by analogy expressions for the mappings of $\mathbf{N}$ onto the bases $\mathbf{E}'$ and $\mathbf{E}''$:

$$\mathbf{N}_{E'} = \tilde{\mathbf{S}} \circ \tilde{\Lambda} \circ \mathbf{N} \circ \Lambda \circ \mathbf{S} = \tilde{\mathbf{S}} \circ \tilde{\Lambda} \circ \mathbf{M} \circ \Lambda \circ \mathbf{S} \circ \tilde{\mathbf{S}} \circ \Lambda \circ \mathbf{S} = \mathbf{M}_{E'} \circ \Lambda_{E'},$$

$$\mathbf{N}_{E''} = \tilde{\mathbf{S}} \circ \tilde{\Lambda} \circ \tilde{\mathbf{M}} \circ \mathbf{N} \circ \mathbf{M} \circ \Lambda \circ \mathbf{S} = \tilde{\mathbf{S}} \circ \tilde{\Lambda} \circ \tilde{\mathbf{M}} \circ \mathbf{M} \circ (\mathbf{M} \circ \Lambda \circ \mathbf{S} \circ \tilde{\mathbf{S}} \circ \tilde{\Lambda} \circ \tilde{\mathbf{M}}) \circ \Lambda \circ \mathbf{M} \circ \Lambda \circ \mathbf{S} = \mathbf{M}_{E''} \circ \Lambda_{E''}.$$

Thus, this theorem permits one to use the formula (2.57) for finding the components of the resulting rotation quaternion in any basis. The components of the quaternions of the component rotations should be expressed in the same basis.

The quaternions $\mathbf{N},\Lambda$ and $\mathbf{M}$ determine the finite rotation vectors of the transformations under consideration. Each quaternion can be written in the form:

$$\left.\begin{aligned}
\Lambda &= \lambda_0 + \sum_{j=1}^{3} \lambda_j \mathbf{i}_j = \lambda_0 + \sum_{j=1}^{3} \lambda_{Ej} \mathbf{e}_j = \\
&= \lambda_0 + \sum_{j=1}^{3} \lambda_{E'j} \mathbf{e}'_j = \lambda_0 + \sum_{j=1}^{3} \lambda_{E''j} \mathbf{e}''_j, \\
\mathbf{M} &= \mu_0 + \sum_{j=1}^{3} \mu_j \mathbf{i}_j = \mu_0 + \sum_{j=1}^{3} \mu_{Ej} \mathbf{e}_j = \\
&= \mu_0 + \sum_{j=1}^{3} \mu_{E'j} \mathbf{e}'_j = \mu_0 + \sum_{j=1}^{3} \mu_{E''j} \mathbf{e}''_j, \\
\mathbf{N} &= \nu_0 + \sum_{j=1}^{3} \nu_j \mathbf{i}_j = \nu_0 + \sum_{j=1}^{3} \nu_{Ej} \mathbf{e}_j = \\
&= \nu_0 + \sum_{j=1}^{3} \nu_{E'j} \mathbf{e}'_j = \nu_0 + \sum_{j=1}^{3} \nu_{E''j} \mathbf{e}''_j.
\end{aligned}\right\} \qquad (2.62)$$

It follows from theorem 2.3 that the rules for finding the products of two transformations $N = M \circ \Lambda$ are preserved for any projections of the vector parts of quaternions onto the same basis. The latter can be interpreted as the possibility of using the unit vectors of the bases **E**,**E'** and **E"** (which should be the case, since the hypercomplex basis $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ can coincide with any three-dimensional basis) as basis vectors of the hypercomplex space **H**. On the other hand, since the operation of multiplying $N = M \circ \Lambda$ is invariant relative to the transformation of the basis, it has a vector meaning. /72

We obtain the vector formula for the summation of the finite rotations. By using the representation (1.9) for multiplying quaternions and introducing, by analogy to (2.51) and (2.52), the finite rotation vectors:

$$\theta_\lambda = 2\frac{\lambda}{\lambda_0}, \quad \theta_\mu = 2\frac{\mu}{\mu_0}, \quad \theta_\nu = 2\frac{\nu}{\nu_0}.$$

we obtain:

$$N = \nu_0 + \nu = M \circ \Lambda = (\mu_0 + \mu) \circ (\lambda_0 + \lambda) = \\ = \mu_0\lambda_0 + \mu\lambda_0 + \lambda\mu_0 - \mu \cdot \lambda + \mu \times \lambda.$$

We separate the scalar and vector parts of the obtained expression:

$$\nu_0 = \mu_0\lambda_0 - \mu \cdot \lambda, \quad \nu = \lambda_0\mu + \mu_0\lambda + \mu \times \lambda.$$

Obviously:

$$\frac{\nu_0}{\lambda_0\mu_0} = 1 - \frac{\mu}{\mu_0} \cdot \frac{\lambda}{\lambda_0} = 1 - \frac{1}{4}\theta_\mu \cdot \theta_\lambda, \quad \frac{\nu}{\nu_0} = \frac{1}{2}\theta_\nu,$$
$$\frac{\nu}{\lambda_0\mu_0} = \frac{\lambda}{\lambda_0} + \frac{\mu}{\mu_0} + \frac{\mu}{\mu_0} \times \frac{\lambda}{\lambda_0} = \frac{1}{2}\left(\theta_\lambda + \theta_\mu + \frac{1}{2}\theta_\mu \times \theta_\lambda\right).$$

By taking the ratio of these expressions, we obtain the vector formula for the summation of finite rotations:

$$\theta_\nu = \frac{1}{1 - \frac{1}{4}\theta_\lambda \cdot \theta_\mu}\left(\theta_\lambda + \theta_\mu + \frac{1}{2}\theta_\mu \times \theta_\lambda\right). \qquad (2.63)$$

Let us further consider three successively performed orthogonal transformations. If the transition from the basis **E** to the basis **E'** is determined by the quaternion **Λ**, from **E'** to **E"** — by the quaternion **M**, and from **E"** to **E"'** — by the quaternion **P**, then the quaternion of the resultant transformation **N** converting /73 the basis **E** into the basis **E"'** is determined by the product (cf. paragraph 1.3):

$$\mathbf{N}=\mathbf{P}\circ\mathbf{M}\circ\mathbf{\Lambda}. \tag{2.64}$$

One can show by analogy to the preceding that the operator multiplication formula (2.64) is valid for the mappings **N,P,M,Λ** for any given coordinate system.

As an example, let us project the equality (2.64) onto the basis **E**. Obviously, we obtain, by analogy to (2.58) - (2.61):

$$\mathbf{N}_E=\tilde{\boldsymbol{S}}\circ\mathbf{N}\circ\boldsymbol{S},\quad \mathbf{P}_E=\tilde{\boldsymbol{S}}\circ\mathbf{P}\circ\boldsymbol{S},\quad \mathbf{M}_E=\tilde{\boldsymbol{S}}\circ\mathbf{M}\circ\boldsymbol{S},$$
$$\mathbf{\Lambda}_E=\tilde{\boldsymbol{S}}\circ\mathbf{\Lambda}\circ\boldsymbol{S},$$
$$\mathbf{N}_E=\tilde{\boldsymbol{S}}\circ\mathbf{P}\circ\mathbf{M}\circ\mathbf{\Lambda}\circ\boldsymbol{S}=(\tilde{\boldsymbol{S}}\circ\mathbf{P}\circ\boldsymbol{S})\circ(\tilde{\boldsymbol{S}}\circ\mathbf{M}\circ\boldsymbol{S})\circ(\tilde{\boldsymbol{S}}\circ\mathbf{\Lambda}\circ\boldsymbol{S})=$$
$$=\mathbf{P}_E\circ\mathbf{M}_E\circ\mathbf{\Lambda}_E$$

In exactly the same way, one can extend the multiplication theorem 2.3 to an arbitrary number of quaternions specifying a number of successive transitions.

The rule for finding the resulting rotation from the known components has a disadvantage in that it is necessary to project the quaternions onto the same coordinate system. In this connection, this rule does not permit one to relate the Rodriguez-Hamilton parameters of the resulting and the component transformations. Actually, by projecting the equality (2.56) onto any of the systems **E,E'** and **E"**, we will obtain corresponding mappings of the quaternions of which only a certain two will be eigenquaternions composed of the Rodriguez-Hamilton parameters.

The quaternion whose components are the Rodriquez-Hamilton parameters has a peculiar meaning. This is explained by the fact that the kinematic equations which determine the relation of the derivatives of the kinematic parameters to the magnitudes of the angular velocity are written for the projections of the finite rotation vector on the initial (fixed) and final (body) coordinate systems. In connection with this, it is these parameters which are used for the applications.

The structure developed here for projecting a quaternion onto various coordinate systems permits one to prove the following theorem which determines the relation of the resultant quaternion $\mathbf{N}^*$ with the components of $\mathbf{\Lambda}^*$ and $\mathbf{M}^*$. /74

Theorem 2.4. Let $\mathbf{\Lambda}^*, \mathbf{M}^*$ and $\mathbf{N}^*$ be eigenquaternions of the first, second and resultant rotations, respectively. Then, the components of the resultant rotation quaternion $\mathbf{N}^*$ are determined by the quaternion components of the first $\mathbf{\Lambda}^*$ and second $\mathbf{M}^*$ rotations from the rule of quaternion multiplication (1.5). The factors are taken in reverse formula:

$$\mathbf{N}^* = \mathbf{\Lambda}^* \circ \mathbf{M}^*. \tag{2.65}$$

Proof: By expressing the quaternions $\mathbf{N}^*$, $\mathbf{\Lambda}^*$ and $\mathbf{M}^*$ using the relations (2.58), (2.60) and (2.61), we obtain formula (2.65) at once:

$$\mathbf{N}^* = \mathbf{N}_E = \tilde{\mathbf{S}} \circ \mathbf{N} \circ \mathbf{S} = \tilde{\mathbf{S}} \circ \mathbf{M} \circ \mathbf{\Lambda} \circ \mathbf{S} =$$
$$= \tilde{\mathbf{S}} \circ \mathbf{\Lambda} \circ \mathbf{S} \circ (\tilde{\mathbf{S}} \circ \tilde{\mathbf{\Lambda}} \circ \mathbf{M} \circ \mathbf{\Lambda} \circ \mathbf{S}) = \mathbf{\Lambda}_E \circ \mathbf{M}_E = \mathbf{\Lambda}^* \circ \mathbf{M}^*.$$

By performing the quaternion multiplication (2.65) and equating the components of the basis vectors, we find:

$$\left.\begin{aligned} \nu_0^* &= \lambda_0^* \mu_0^* - \lambda_1^* \mu_1^* - \lambda_2^* \mu_2^* - \lambda_3^* \mu_3^*, \\ \nu_1^* &= \lambda_0^* \mu_1^* + \lambda_1^* \mu_0^* + \lambda_2^* \mu_3^* - \lambda_3^* \mu_2^*, \\ \nu_2^* &= \lambda_0^* \mu_2^* + \lambda_2^* \mu_0^* + \lambda_3^* \mu_1^* - \lambda_1^* \mu_3^*, \\ \nu_3^* &= \lambda_0^* \mu_3^* + \lambda_3^* \mu_0^* + \lambda_1^* \mu_2^* - \lambda_2^* \mu_1^*. \end{aligned}\right\} \tag{2.66}$$

By comparing the relations (2.57) and (2.66), one can verify that the signs of the last two terms of the vector part of the quaternion have changed. We also note that the formulae, (2.66) can be obtained from the relations (2.57), if they are written in the projections , for example, on the basis E'. It is then necessary to project the components of the vector N onto the basis E or E" using a direction cosine matrix of the type (2.35).

Theorem 2.4 remains valid also for the case of three successively performed transformations defined respectively by the quaternions Λ,M and P In this case, by taking into account the relations (2.58) and (2.60) and also the fact that the resultant quaternion N is determined by the relation (2.64), we find: / 75

$$N^* = N_E = N_{E'}, \quad \Lambda^* = \Lambda_E = \Lambda_{E'}, \quad M^* = M_{E'} = M_{E''}.$$

$$P^* = P_{E''} = \tilde{S} \circ \tilde{\Lambda} \circ \tilde{M} \circ P \circ M \circ \Lambda \circ S =$$

$$= \tilde{S} \circ \tilde{\Lambda} \circ \tilde{M} \circ \tilde{P} \circ P \circ P \circ M \circ \Lambda \circ S = P_{E'''}$$

and further:

$$N^* = \tilde{S} \circ P \circ M \circ \Lambda \circ S =$$

$$= (\tilde{S} \circ \Lambda \circ S) \circ (\tilde{S} \circ \tilde{\Lambda} \circ M \circ \Lambda \circ S) \circ (\tilde{S} \circ \tilde{\Lambda} \circ \tilde{M} \circ P \circ M \circ \Lambda \circ S) =$$

$$= \Lambda^* \circ M^* \circ P^*.$$

Since any two rotations written in the form (2.65) can be combined into one, then the successive application of this multiplication formula permits one to obtain at once the multiplication formula for three rotations, which was just now proven. Obviously, one can extend this theorem to the case of an arbitrary number of successively performed transformations by the method of induction.

This study shows that the eigenquaternions of the transformations with their addition are also multiplied according to the quaternion multiplication rules. However, this form of transformation addition does not yet have a vector meaning. Nevertheless, it is this addition form which is valid, as will

be shown below, for the addition of relative motion. The property of quaternions to describe three-dimensional spatial motion in a more complete manner is reflected in this.

In conclusion, we will consider the well-known theorem on the commutivity of finite rotations.

Theorem 2.5. The resultant transformation will be exactly the same as if firstly the second rotation is performed and then the first about an axis transformed by the preceding rotation.

The proof of this theorem is obtained at once from the quaternion multiplication formula. If the quaternion Λ defines the first rotation, and the quaternion M — is the second, then the resultant rotation will be defined by the quaternion:

$$N = M \circ \Lambda = M \circ \Lambda \circ \tilde{M} \circ M = \Lambda_M \circ M,$$

where Λ_M denotes the quaternion $M \circ \Lambda \circ \tilde{M}$, which is obtained by transforming the quaternion Λ of the rotation operation defined by the quaternion M. 76

One can see, in addition to this, that:

$$N = M \circ \Lambda = \Lambda \circ \tilde{\Lambda} \circ M \circ \Lambda = \Lambda \circ M_{\tilde{\Lambda}},$$

where $M_{\tilde{\Lambda}}$ is the quaternion whose vector part is transformed by the inverse transformation $\tilde{\Lambda}$. Hence, it follows that rotations can be permuted thusly. First, the second rotation is made about an axis transformed by the inverse first rotation, and then the first rotation is performed about the initial axis.

One can extend the theorem about the commutivity of rotation to the case of three rotations. Let the first rotation be defined by the quaternion Λ, the second — by the quaternion M, the

third — by the quaternion **P**. Then the resultant rotation will be specified by the quaternion:

$$\mathbf{N} = \mathbf{P} \circ \mathbf{M} \circ \Lambda = \mathbf{P} \circ \mathbf{M} \circ \Lambda \circ (\tilde{\mathbf{M}} \circ \tilde{\mathbf{P}} \circ \mathbf{P} \circ \mathbf{M}) \circ (\tilde{\mathbf{P}} \circ \mathbf{P}) =$$
$$= (\mathbf{P} \circ \mathbf{M} \circ \Lambda \circ \tilde{\mathbf{M}} \circ \tilde{\mathbf{P}}) \circ (\mathbf{P} \circ \mathbf{M} \circ \tilde{\mathbf{P}}) \circ \mathbf{P} = \Lambda_{\mathbf{PM}} \circ \mathbf{M}_{\mathbf{P}} \circ \mathbf{P},$$

where:

$$\Lambda_{\mathbf{PM}} = (\mathbf{P} \circ \mathbf{M}) \circ \Lambda \circ (\tilde{\mathbf{P}} \circ \tilde{\mathbf{M}}), \quad \mathbf{M}_{\mathbf{P}} = \mathbf{P} \circ \mathbf{M} \circ \tilde{\mathbf{P}},$$

i.e., the same resultant rotation is obtained if, first, the third rotation is performed, then the second about an axis is transformed by the preceding rotation, then the first rotation about an axis transformed by the preceding two.

It is not difficult to formally extend this theorem to the case of an arbitrary number of successively performed transformations.

§ 2.7. RELATION OF THE RODRIGUEZ-HAMILTON PARAMETERS WITH OTHER KINEMATIC PARAMETERS

The eigenquaternion of the transformation has components which are the Rodriguez-Hamilton parameters. The relation between the components of the quaternion specifying the rotation transformation and the direction cosines, which are elements of an orthogonal transformation matrix, were investigated in detail in the second paragraph of this chapter.

The correspondence of the orthogonal transformation in the form (2.18), (2.19), (2.34) to the rotation transformation (2.33), (2.35) was established. In accordance with this, the correspondence of the elements of the quaternion to the direction cosines defining the basis transformation in the form (2.13) or (2.40) and the vector component transformation (2.15) or (2.17) /77

was established. The elements of the transformation matrix $B = A^T$ are expressed in terms of quaternion components by the relations (2.34) - (2.37).

We will establish the relation of the transformation quaternion with the other kinematic parameters.

I _ The Euler and Krylov Angles

A sequence of three rotations about different noncoinciding axes specifies, as a whole, an arbitrary transformation. Such a transformation is called rotation by the Euler angles. With the rotation about the coordinate axis, two sequences of rotations are distinguished: rotation by the Euler angles and rotation by the Krylov angles.

We will consider first the Euler sequence of three plane rotations carrying the basis I into the basis E, accomplished about the coordinate axis (Figure 2.1). Let the first rotation occur about the axis i_3 by the precession angle φ. The second rotation — about the axis i_1 by the nutation angle ψ, and the third rotation — about the axis A by the rotation angle ϑ. The transformation matrix A specifying the transition from the coordinate system I to the system E in the form (2.13) is obtained in the form of the product of the three matrices of the successive rotations:

$$A = \begin{vmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & -\sin\psi & \cos\psi \\ 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \end{vmatrix} \times$$

$$\times \begin{vmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

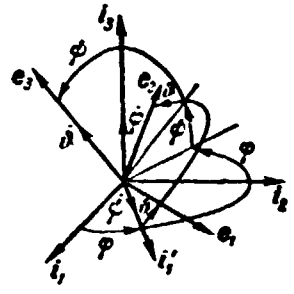

Figure 2.1.

Performing the multiplication, we obtain: / 78

$$A=\begin{vmatrix} \cos\vartheta\cos\varphi-\sin\vartheta\sin\varphi\cos\psi & \cos\vartheta\sin\varphi+\sin\vartheta\cos\varphi\cos\psi & \sin\vartheta\sin\psi \\ -\sin\vartheta\cos\varphi-\cos\vartheta\cos\psi\sin\varphi & -\sin\varphi\sin\vartheta+\cos\vartheta\cos\varphi\cos\psi & \cos\vartheta\sin\psi \\ \sin\varphi\sin\psi & -\sin\psi\cos\varphi & \cos\psi \end{vmatrix}. \qquad (2.67)$$

In an analogous manner, one can find the quaternion of the resultant transformation from the quaternions of each plane rotation. In accordance with the fundamental theorem 1.1, the quaternion of a plane rotation by the angle ϑ equals $\cos\frac{\vartheta}{2}+\zeta\sin\frac{\vartheta}{2}$, where ζ is the unit vector of the rotation axis. To find the resultant quaternion, one would have to represent each of the component quaternions in one basis, in accordance with theorem 2.3. However, two rotation vectors as a minimum would then have taken place about the coordinate axes and their expressions would be rather complicated. It is more convenient to use the results of theorem 2.4.

Let Λ be the quaternion of the first rotation performed about the axis i_3 by the angle φ. Then obviously:

$$\Lambda^*=\cos\frac{\varphi}{2}+i_3\sin\frac{\varphi}{2}.$$

The second rotation is performed about the axis i_1' by the angle ψ; the eigenquaternion M^* of this rotation equals:

$$M^*=\cos\frac{\psi}{2}+i_1\sin\frac{\psi}{2}.$$

The third rotation is performed about the axis e_3 by the angle

The quaternion of this rotation:

$$\mathbf{P}^* = \cos\frac{\vartheta}{2} + i_3 \sin\frac{\vartheta}{2}.$$

According to theorem 2.4, the resultant eigenquaternion $\mathbf{N}^*$ will be equal to the product:

$$\begin{aligned}\mathbf{A}^* \circ \mathbf{M}^* \circ \mathbf{P}^* = \\ = \left(\cos\frac{\varphi}{2} + i_3 \sin\frac{\varphi}{2}\right) \circ \left(\cos\frac{\psi}{2} + i_1 \sin\frac{\psi}{2}\right) \circ \left(\cos\frac{\vartheta}{2} + i_3 \sin\frac{\vartheta}{2}\right) = \\ = \cos\frac{\psi}{2}\cos\frac{\varphi+\vartheta}{2} + i_1 \sin\frac{\psi}{2}\cos\frac{\varphi-\vartheta}{2} + \\ + i_2 \sin\frac{\psi}{2}\sin\frac{\varphi-\vartheta}{2} + i_3 \cos\frac{\psi}{2}\sin\frac{\varphi+\vartheta}{2}.\end{aligned} \tag{2.68}$$

Hence, the following relation of the quaternion components with the Euler angles results:

$$\left.\begin{aligned} \nu_0^* &= \cos\frac{\psi}{2}\cos\frac{\varphi+\vartheta}{2}, & \nu_1^* &= \sin\frac{\psi}{2}\cos\frac{\varphi-\vartheta}{2}, \\ \nu_2^* &= \sin\frac{\psi}{2}\sin\frac{\varphi-\vartheta}{2}, & \nu_3^* &= \cos\frac{\psi}{2}\sin\frac{\varphi+\vartheta}{2}. \end{aligned}\right\} \tag{2.69}$$

By using these expressions for the components of the quaternion $\mathbf{N}^*$, one can obtain with the help of the rotation operations the formulae for the transition of the basis I to the basis **E** [for example, in the form (2.44)] and the formulae for finding the components of a fixed vector [like (2.48)]. One can verify that the elements of the transition matrix of the form (2.35) with the consideration of (2.69) coincide with the matrix $\mathbf{A}^T$ obtained above (2.67).

We will now consider the sequence of three rotations by the Krylov angles (Figure 2.2) also performed about the coordinate axes of the transformed basis. The first rotation is performed about the axis i_3 by the course angle φ. The second rotation is produced about the axis i'_2 by the bank angle ψ, and the third about the axis e_1 by the pitch angle ϑ. The transformation matrix A specifying the transition from the coordinate system I to the system **E** is obtained by multiplying the three matrices of t plane rotations:

$$\mathbf{A} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{Vmatrix} \begin{Vmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{Vmatrix} \begin{Vmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

By performing the matrix multiplication, we obtain:

$$A=\begin{vmatrix} \cos\varphi\cos\psi & \cos\psi\sin\varphi & -\sin\psi \\ \sin\vartheta\sin\psi\cos\varphi-\cos\vartheta\sin\varphi & \sin\vartheta\sin\psi\sin\varphi+\cos\vartheta\cos\varphi & \sin\vartheta\cos\psi \\ \cos\vartheta\sin\psi\cos\varphi+\sin\vartheta\sin\varphi & \cos\vartheta\sin\psi\sin\varphi-\sin\vartheta\cos\varphi & \cos\vartheta\cos\psi \end{vmatrix}. \qquad (2.70)$$

We further obtain the rotation transformation quaternion accomplishing the transition from the basis I to the basis E.
By analogy to the preceding case, we first find the eigenquaternions of the successive plane rotations. The first rotation performed about the axis i_3 by the angle φ corresponds to the quaternion: /80

$$A^*=\cos\frac{\varphi}{2}+i_3\sin\frac{\varphi}{2}.$$

The second rotation about the axis i'_2 by the angle ψ corresponds to the quaternion:

$$M^*=\cos\frac{\psi}{2}+i_2\sin\frac{\psi}{2}.$$

and the third rotation (about the axis e_1 by the angle ϑ) corresponds to the quaternion:

$$P^*=\cos\frac{\vartheta}{2}+i_1\sin\frac{\vartheta}{2}.$$

In accordance with theorem 2.4, we have the resultant rotation quaternion:

$$N^*=A^*\circ M^*\circ P^*= \qquad (2.71)$$
$$=\left(\cos\frac{\varphi}{2}+i_3\sin\frac{\varphi}{2}\right)\circ\left(\cos\frac{\psi}{2}+i_2\sin\frac{\psi}{2}\right)\circ\left(\cos\frac{\vartheta}{2}+i_1\sin\frac{\vartheta}{2}\right).$$

By performing the quaternion multiplication, we obtain the following values for its components as a function of the Krylov angles:

$$\left.\begin{aligned} v_0^*&=\cos\frac{\varphi}{2}\cos\frac{\psi}{2}\cos\frac{\vartheta}{2}+\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\sin\frac{\vartheta}{2},\\ v_1^*&=\cos\frac{\varphi}{2}\cos\frac{\psi}{2}\sin\frac{\vartheta}{2}-\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\cos\frac{\vartheta}{2},\\ v_2^*&=\sin\frac{\psi}{2}\cos\frac{\varphi}{2}\cos\frac{\vartheta}{2}+\sin\frac{\varphi}{2}\cos\frac{\psi}{2}\sin\frac{\vartheta}{2},\\ v_3^*&=\sin\frac{\varphi}{2}\cos\frac{\psi}{2}\cos\frac{\vartheta}{2}-\sin\frac{\psi}{2}\cos\frac{\varphi}{2}\sin\frac{\vartheta}{2}. \end{aligned}\right\} \qquad (2.72)$$

The values obtained for the quaternion permit one to write the orthogonal transformation operation in the form of the rotation operation [for example, in the form (2.44), (2.48)]. The reader can verify that the elements of the rotation operation matrix of the form (2.35) coincide with the elements of the matrix $\mathbf{A}^T$ (2.70). /81

With the rotation axes and rotation angles given, the sequence of three Euler rotations, completely defines the transformation. On the other hand, if the required transformation and the rotation axes and their sequence are specified, then the Euler rotation angles are uniquely defined by these. Actually, the transformation is defined by the components v_i of the transformation quaternion. One can see that the formulae (2.69) or (2.72) are uniquely resolved relative to the magnitudes of the rotation angles.

Rotation by the Euler angles is a rotation about the coordinate axes (of course, successively transformed). Let us pose the question: can an arbitrarily specified transformation be performed by three rotations relative to arbitrary axes and, if it is impossible, what are the conditions on the arrangement of the rotation axes? The application of quaternions permits one to investigate this question rather completely and thoroughly.

Let us consider an arbitrary transformation which is defined by the given quaternion **N**. By representing the quaternion **N** by an arc on the unit sphere one can see (cf. paragraph 1.2) that this arc occupies a free position on the great circle defined by the vector part $\boldsymbol{\nu}$ of the quaternion and its length equals half the rotation angle of the quaternion **N**. Let the same transformation be accomplished by rotations about several given axes. Each axis of rotation defines on the sphere a great circle on which the arc of the corresponding quaternion composing the

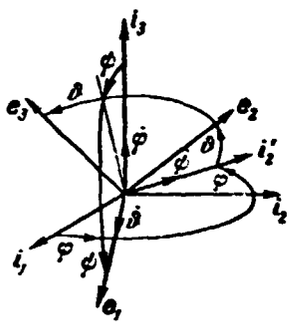

Figure 2.2.

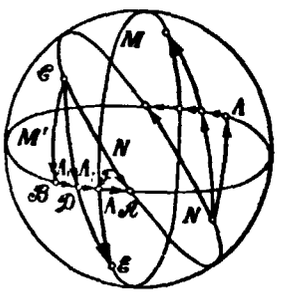

Figure 2.3.

resultant transformation should be placed.

We will first consider the case when two axes of rotations composing a given transformation are specified. These axes define the circles of Λ and $\mathbf{M}$ on the sphere (Figure 2.3). Obviously, the given quaternion $\mathbf{N}$ can be represented by the arc $\mathscr{AC}$ in the form of the product of the two quaternions represented by the arcs $\mathscr{AB}$ and $\mathscr{BC}$.

$$\mathbf{N}=\Lambda\circ\mathbf{M}',$$

The arc $\mathbf{M}'$ is obtained from the "resolved" arc $\mathbf{M}$ (i.e., $\mathscr{EF}$) / 82 by means of its transfer by the rotation operation. Actually, we find from the triangles $\mathscr{DBC}$ and $\mathscr{EFD}$:

$$\mathscr{ED}=\mathscr{EF}+\mathscr{FD}=\mathscr{DC}=\mathscr{DB}+\mathscr{BC},$$

or denoting the arcs $\mathscr{ED}=\mathscr{DC}$ and $\mathscr{FD}=\mathscr{DB}$ by the quaternions $\mathbf{P}$ and Λ respectively, we have:

$$\mathbf{P}=\mathbf{M}\circ\Lambda_1=\Lambda_1\circ\mathbf{M}',$$

hence:

$$\mathbf{M}'=\tilde{\Lambda}_1\circ\mathbf{M}\circ\Lambda_1.$$

In accordance with this, we have the following expression for the given quaternion N in terms of the "resolved" directions of rotation:

$$\mathbf{N}=\mathbf{\Lambda}\circ\mathbf{M}'=\mathbf{\Lambda}\circ\tilde{\mathbf{\Lambda}}_1\circ\mathbf{M}\circ\mathbf{\Lambda}_1.$$

Since the rotations $\mathbf{\Lambda}$ and $\mathbf{\Lambda}_1$ are produced about one axis, then the transformation $\mathbf{\Lambda}\circ\tilde{\mathbf{\Lambda}}_1$ represents one rotation about this same axis, which we denote by $\mathbf{\Lambda}_2$ as a result of which:

$$\mathbf{N}=\mathbf{\Lambda}_2\circ\mathbf{M}\circ\mathbf{\Lambda}_1. \tag{2.73}$$

Obviously, this is the Euler sequence of rotations of the form (2.68). We note that the addition of the quaternion arcs can be produced at the two points of intersection of the arc N with the arcs $\mathbf{\Lambda}$ and M.

One such addition is shown in Figure 2.3 on the opposite side of the sphere, which also reduces to the relation (2.73). The addition beginning from the point of intersection of the arcs N and M is presented in Figure 2.4. We have, in accordance with Figure 2.4:

$$\mathbf{N}=\mathbf{\Lambda}'\circ\mathbf{M},\quad \mathbf{\Lambda}'\circ\mathbf{M}_1=\mathbf{M}_1\circ\mathbf{\Lambda},$$

from which:

$$\mathbf{N}=\mathbf{M}_1\circ\mathbf{\Lambda}\circ\tilde{\mathbf{M}}_1\circ\mathbf{M}=\mathbf{M}_1\circ\mathbf{\Lambda}\circ\mathbf{M}_2,$$

which obviously corresponds to the other sequence of Euler rotations.

One can see that the requirement of the "perpendicularity" of the arcs of the component quaternions (i.e., the "resolved" rotation axes) is not necessary. However, it follows from the examination of Figure 2.5 that, for the presented arrangement of the great circles of the component rotations $\mathbf{\Lambda}$ and M, not every

rotation N can be represented by their product. Actually, the rotation transformation around the axis Λ can "carry" the arc M, not to any point on the sphere, but only to the region bounded by the small circles $\mathscr{L}$ and $\mathscr{E}$. We also have an analogous condition for the other sequence of rotations, which defines an analogous region bounded by the planes $\mathscr{L}'$ and $\mathscr{E}'$ (shown dashed in Figure 2.5). We note that the maximum length of the arcs of the quaternions N' and N" with both sequences can be different. Consequently, one can, with the help of the product of the quaternions M and Λ, represent the arc N', which is put into the region bounded by the planes $\mathscr{L}$ and $\mathscr{E}$ or $\mathscr{L}'$ and $\mathscr{E}'$. Hence, it is seen that the arbitrary quaternion with an angle $\theta_v \in \left[0, \frac{\pi}{2}\right]$ can be expressed by the Euler sequence of rotations only when the axes of the component rotations are perpendicular. However, when $\theta_v < \frac{\pi}{2}$, it is possible also to use nonperpendicular rotations. /83

We will consider the representation of the quaternion N with the help of the product of three different rotations. Let the three directions of the rotations defined by the great circles Λ, M and P on the sphere (Figure 2.6) be given. The arcs of the quaternions composing the given quaternion N will be arranged on the circles. The arc of the quaternion N is placed on the corresponding great circle. If the sequence of rotations [for example, Λ, M, P, as in this case] is given, then the geometry for adding the arcs on the sphere is uniquely specified by this. Actually, we find from the spherical triangle $\mathscr{CBA}$: /84

$$\mathbf{N} = \Lambda \circ \mathbf{K},$$

and from the triangle $\mathscr{DEF}$, such that $\mathscr{DE} = \mathscr{CB}$, we have:

$$\mathbf{K} = \mathbf{M} \circ \mathbf{P},$$

from which:

$$\mathbf{N} = \Lambda \circ \mathbf{M} \circ \mathbf{P}, \qquad (2.74)$$

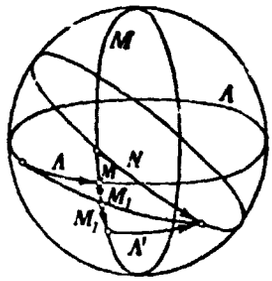

Figure 2.4.

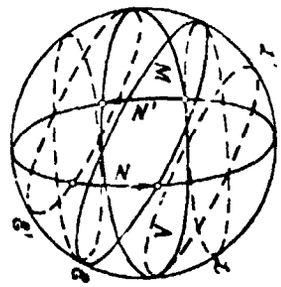

Figure 2.5.

i.e., the sequence of the factors corresponding to the rotations by the Krylov angles (2.71) are obtained. One can see that the position of the great circle of the quaternion **K** is fixed by the point $\mathscr{C}$ (by the end of the arc of the quaternion **N**) and by the equality condition of the arcs $\mathscr{DE}$ and $\mathscr{CA}$. Exactly the same triangle is on the opposite side of the sphere. The other geometry for adding the arcs will correspond to the other sequence of rotations.

It is seen from Figure 2.6 that the requirement of mutual perpendiculiarity of the component rotations is not necessary. Nevertheless, this resolution is not possible for every position of the component arcs. Analysis shows that a sufficient condition for performing the resolution is the requirement that the arc of the quaternion **K** (the arc $\mathscr{CA}$) not be greater than the maximum possible distance between the corresponding points of the arcs of the quaternions **M** and **P**. Thus, for example, one can see that for the closed circles of **M** and P the case can take place when the arc $\mathscr{DE}$ will always be less than the arc $\mathscr{CA}$, i.e., the resolution cannot be fulfilled. Even if only two of the three rotation axes are perpendicular, then the resolution is always possible. Then with the continuous approach of one great circle to another, the sequence of rotations approaches the Euler sequence.

II. The Cayley-Klein Parameters

The Cayley-Klein parameters are complex combinations of the components of a quaternion. They are introduced into the kinematics of rigid body rotation either by means of the stereographic projection of the sphere onto the plane and the specification of a linear fractional transformation of the plane onto itself [31,44] or, with the help of the isomorphic operation of the linear transformation specified by a unitary matrix [16]. It is more convenient to use the second way to establish the relation of a quaternion with the Cayley-Klein parameters. We will briefly discuss the essence of this method here. A more complete discussion will be found by the reader in work [16]. /85

Let the complex two-dimensional space uv be given and its linear transformation:

$$u' = \alpha u + \beta v, \quad v' = \gamma u + \delta v,$$

determined by the matrix:

$$U = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}. \tag{2.75}$$

be specified. The matrix which is the complex conjugate and transposed relative to the matrix U (the complex conjugate of a quantity will be denoted by an asterisk), i.e.:

$$U^{*} = \begin{vmatrix} \alpha^{*} & \gamma^{*} \\ \beta^{*} & \delta^{*} \end{vmatrix} = U^{+},$$

is called the Hermitian conjugate matrix of the matrix U. We will consider transformations for which the following conditions are satisfied:

$$\left.\begin{aligned} UU^{+} &= U^{+}U = 1, \\ \det U &= 1 . \end{aligned}\right\} \tag{2.76}$$

(here $1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ is the unit matrix).

Matrices satisfying the first condition of (2.76) are called unitary matrices. The conditions (2.76) lead to the following relations which must be satisfied by the elements α,β,δ and γ.

$$\left.\begin{aligned} \alpha^*\alpha+\beta^*\beta&=1,\\ \gamma^*\gamma+\delta^*\delta&=1,\\ \alpha^*\gamma+\beta^*\delta&=0,\\ \alpha\delta-\beta\gamma&=1. \end{aligned}\right\} \tag{2.77}$$

Here the first three equalities are the unitarity conditions of the matrix U, and the last corresponds to the requirement of a unit determinant of the matrix. From the third equality of (2.77) we obtain the relation: / 86

$$\frac{\delta}{\gamma}=-\frac{\alpha^*}{\beta^*},$$

which with substitution into the fourth leads to the expression:

$$-(\alpha\alpha^*+\beta\beta^*)\frac{\gamma}{\beta^*}=1.$$

With consideration of the first equality of (2.77), we obtain the following conditions on the elements of the matrix U, to which the conditions (2.76) and (2.77) are reduced

$$\left.\begin{aligned} \gamma&=-\beta^*,\quad \alpha^*=\delta,\\ \alpha\alpha^*+\beta\beta^*&=1. \end{aligned}\right\} \tag{2.78}$$

The four elements α,β,γ and δ satisfying the condition (2.78) are called the Cayley-Klein parameters. The linear transformation matrix (2.75) contains eight quantities in all, since each of its elements is complex.

The equalities (2.77) contain the following conditions: the first two equalities are real; the third is complex. Thus, the requirement that the transformation matrix be unitary contains four conditions. The fifth condition is the requirement on the determinant of the matrix U. Thus, the matrix U satisfying the conditions (2.76) and (2.77) contains only three independent

parameters, i.e., just the number which is necessary for describing the orientation of a rigid body in three-dimensional space.

Let us consider a matrix operation in two-dimensional complex space:

$$R=\begin{vmatrix} r_3 & r_1-ir_2 \\ r_1+ir_2 & -r_3 \end{vmatrix}. \tag{2.79}$$

whose real components r_1, r_2, r_3 will be interpreted as the coordinates of the vector r in three-dimensional space. Let the matrix R be subjected to the following transformation by means of the unitary matrix U: / 87

$$R'=URU^{+}. \tag{2.80}$$

The relation (2.80) describes a similar transformation of the matrix R which occurs when the space uv experiences the unitary transformation (we note that $U^{-1} = U^{+}$, which follows from (2.76). Since the matrix R is Hermitian (i.e., the Hermitian conjugate of the matrix before the matrix itself $R^{+} = R$) and its trace equals zero, then the matrix R'must also be Hermitian and have a trace equaling zero (the latter follows from the fact that a similar transformation does not change either of these matrix properties). Hence, it follows that the matrix R' must have the form:

$$R'=\begin{vmatrix} r'_3 & r'_1-ir'_2 \\ r'_1+ir'_2 & -r'_3 \end{vmatrix}.$$

where r'_1,r'_2 and r'_3 are real numbers. Since the determinant of the matrix also does not change with a similar transformation, then:

$$\det R=-(r_1^2+r_2^2+r_3^2)=\det R'=-[(r'_1)^2+(r'_2)^2+(r'_3)^2]. \tag{2.81}$$

The relation (2.81) is the transformation orthogonality condition. In accordance with this, one can consider the operation (2.80) as an operation transforming the vector r into the new vector r'. By performing matrix multiplication in relation

(2.80), we obtain the following relation between the components r'_i and the components r_i $(i = 1, 2, 3)$:

$$\begin{vmatrix} r'_3 & r'_1 - ir'_2 \\ r'_1 + ir'_2 & -r'_3 \end{vmatrix} =$$

$$= \begin{vmatrix} (\alpha\delta + \beta\gamma) r_3 - \alpha\gamma (r_1 - ir_2) + \beta\delta (r_1 + ir_2) & -2\alpha\beta r_3 + \alpha^2 (r_1 - ir_2) - \beta^2 (r_1 + ir_2) \\ 2\gamma\delta r_3 - \gamma^2 (r_1 - ir_2) + \delta^2 (r_1 + ir_2) & -(\alpha\delta + \beta\gamma) r_3 + \alpha\gamma (r_1 - ir_2) - \beta\delta (r_1 + ir_2) \end{vmatrix}.$$

If the relation between the components r'_i and r_i is represented in the form of relation (2.18), then the transformation matrix, in accordance with the obtained equality, will equal: /88

$$A^T = B = \begin{vmatrix} \frac{1}{2}(\alpha^2 - \gamma^2 + \delta^2 - \beta^2) & \frac{i}{2}(\gamma^2 - \alpha^2 + \delta^2 - \beta^2) & \gamma\delta - \alpha\beta \\ \frac{i}{2}(\alpha^2 + \gamma^2 - \beta^2 - \delta^2) & \frac{1}{2}(\alpha^2 + \gamma^2 + \beta^2 + \delta^2) & -i(\alpha\beta - \gamma\delta) \\ \beta\delta - \alpha\gamma & i(\alpha\gamma + \beta\delta) & \alpha\delta + \beta\gamma \end{vmatrix}. \quad (2.82)$$

One can show that the orthogonal transformation (2.81) is isomorphic to the operation (2.80) and can develop the formulism for all the transformations related to rigid body motion. However, it is more convenient to establish the relation between the Cayley-Klein parameters and quaternions in order to use the results obtained above. This relation is easily established with the help of the Pauli spin matrices, which are the following three two-dimensional matrices:

$$\sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \sigma_2 = \begin{vmatrix} 0 & i \\ i & 0 \end{vmatrix}, \quad \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}. \quad (2.83)$$

We note that all the spin matrices are Hermitian (i.e., $\sigma_i^+ = \sigma_i$), unitary, and the trace of each equals zero. Together with the unit matrix:

$$1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

The spin matrices form a system of four independent matrices of two-dimensional complex space. Any 2 X 2 matrix in two-dimensional space can be represented in the form of a linear combination of these four matrices:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{1}{2}(b+c)\,p_1 + \frac{i}{2}(b-c)\,p_2 + \frac{1}{2}(a-d)\,p_3 + \frac{1}{2}(a+d)\,1. \qquad (2.84)$$

If the trace of this matrix equals zero, then the last term in (2.84) vanishes. In addition, if the matrix is Hermitian (i.e., a, d are real numbers and b = c*), then the coefficients of the spin matrices in the expansion (2.84) will be real numbers. The Pauli matrices permit one to represent the operator R in the following form: /89

$$R = \begin{vmatrix} r_3 & r_1 - ir_2 \\ r_1 + ir_2 & -r_3 \end{vmatrix} = r_1 p_1 + r_2 p_2 + r_3 p_3. \qquad (2.85)$$

Thanks to this, the matrix operator R, which is set in correspondence with the vector **r**, is represented in the form of a resolution of the vector along "basis vectors" of three-dimensional space. The Pauli spin matrices emerge as such basis vectors. This itself establishes the relations of two-dimensional complex space and three-dimensional real space.

Let us consider how one can represent the unitary matrix U with the help of spin matrices.

We have, in accordance with (2.84)

$$U = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \frac{1}{2}(\beta+\gamma)\,p_1 + \frac{i}{2}(\beta-\gamma)\,p_2 + \frac{1}{2}(\alpha-\delta)\,p_3 + \frac{1}{2}(\alpha+\delta)\,1 \qquad (2.86)$$

One can see [cf. relations (2.78)] that the coefficients of p_i are imaginary numbers and the coefficient of the unit matrix is a real number. Hence, it follows that the Hermitian conjugate of the matrix U equals:

$$U^+ = \frac{1}{2}(\alpha+\delta)\,1 - \frac{1}{2}(\beta+\gamma)\,p_1 - \frac{i}{2}(\beta-\gamma)\,p_2 - \frac{1}{2}(\alpha-\delta)\,p_3,$$

i.e., the sign of the coefficients of the spin matrices is changed.

We will now show that each spin matrix defines a unit rotation operator about its own axis. Actually, the transformation operations (2.80) for the matrix U equaling, for example:

$$U=\alpha 1+\beta p_1$$

(which occurs when $\alpha=\delta$ and $\beta=\gamma$) leaves the vector having the same direction fixed, i.e.,: /90

$$R=r_1p_1=R'=r_1'p_1.$$

By substituting these quantities into relation (2.80), we obtain:

$$R'=URU^+=(\alpha 1+\beta p_1)r_1p_1(\alpha 1-\beta p_1)=r_1(\alpha^2-\beta^2)p_1=r_1p_1$$

(it is necessary to take the fact into account that $p_i1=1p_i=p_i$ and $p_ip_i=1$).

One can show that if γ_i are the direction cosines of the unit vector, then the unitary matrix:

$$U=1\cos\frac{\Phi}{2}+i(\gamma_1p_1+\gamma_2p_2+\gamma_3p_3)\sin\frac{\Phi}{2} \qquad (2.87)$$

defines the transformation (2.80) leaving fixed the vector $\gamma_1p_1+\gamma_2p_2+\gamma_3p_3$, which will also be the rotation axis (the rotation angle equals Φ). The representation of the unitary matrix (2.86) and the transformation (2.80) become equivalent according to representation of the quaternion (1.11) and rotation operation (1.14). If one takes the fact into account that the spin matrices permitting the "vector" representation of the operators R and U satisfy the rules for multiplying the hypercomplex units:

$$\left.\begin{array}{ll} p_1^2=p_2^2=p_3^2=1, & \\ p_2p_3=-p_3p_2=ip_1, & p_i1=1p_i=p_i, \\ p_3p_1=-p_1p_3=ip_2, & p_21=1p_2=p_2, \\ p_1p_2=-p_2p_1=ip_3, & p_31=1p_3=p_3. \end{array}\right\} \qquad (2.88)$$

Thus, the spin matrices permit one to establish the correspondence between quaternions and unitary matrices. We note, however, that the spin matrices represent somewhat differently the vector matrix (with real coefficients) (2.85) and the transformation operator matrix U (2.86) (the coefficients are imaginary for the vector part). Because of this, the formulae for multiplying the spin matrices (2.88) are not obtained so symmetrically. Nevertheless, the equivalence of the transformation matrix (2.87) to the quaternion shows that the following relations are satisfied:

$$\lambda_0 = \cos\frac{\Phi}{2} = \frac{1}{2}(\alpha + \delta), \qquad \lambda_1 = \gamma_1 \sin\frac{\Phi}{2} = -\frac{i}{2}(\beta + \gamma),$$
$$\lambda_2 = \gamma_2 \sin\frac{\Phi}{2} = \frac{1}{2}(\beta - \gamma), \qquad \lambda_3 = \gamma_3 \sin\frac{\Phi}{2} = -\frac{i}{2}(\alpha - \delta),$$

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where λ_i $(i = 0, 1, 2, 3)$ can, in particular, be the Rodriguez-Hamilton parameters.

With consideration of these relations following from the equality (2.86) one can relate the Cayley-Klein parameters with the components of a quaternion:

$$\left.\begin{aligned} \alpha &= \lambda_0 + i\lambda_3, & \delta &= \lambda_0 - i\lambda_3, \\ \beta &= \lambda_2 + i\lambda_1, & \gamma &= i\lambda_1 - \lambda_2. \end{aligned}\right\} \qquad (2.89)$$

By replacing the unitary matrix in expression (2.86) by the components λ_i, one can reduce the transformation (2.80) to the transformation (2.33). All of the results obtained in paragraphs 2.2 - 2.6 are applicable.

We will note only that the operation (2.80) is isomorphic to the orthogonal transformation in the form (2.18).

In conclusion, by using the relations (2.89) and the equalities (2.69) and (2.72) obtained previously, we will relate the Cayley-Klein parameters with the Euler and Krylov angles.

By sutstituting the quantities ν_i from (2.69) in place of λ_i $(i=0, 1, 2, 3)$, we obtain for the Euler angles:

$$\left.\begin{aligned} \alpha &= \cos\frac{\psi}{2}e^{i\frac{\varphi+\theta}{2}}, \quad \delta = \cos\frac{\psi}{2}e^{-i\frac{\varphi+\theta}{2}}, \\ \beta &= i\sin\frac{\psi}{2}e^{i\frac{\theta-\varphi}{2}}, \quad \gamma = i\sin\frac{\psi}{2}e^{i\frac{\varphi-\theta}{2}}. \end{aligned}\right\} \tag{2.90}$$

By using the relations (2.72), we find the dependence of the Cayley-Klein parameters on the Krylov angles:

$$\left.\begin{aligned} \alpha &= e^{\frac{i\varphi}{2}}\left(\cos\frac{\psi}{2}\cos\frac{\theta}{2} - i\sin\frac{\psi}{2}\sin\frac{\theta}{2}\right), \\ \delta &= e^{-\frac{i\varphi}{2}}\left(\cos\frac{\psi}{2}\cos\frac{\theta}{2} + i\sin\frac{\psi}{2}\sin\frac{\theta}{2}\right), \\ \beta &= e^{-\frac{i\varphi}{2}}\left(\cos\frac{\theta}{2}\sin\frac{\psi}{2} + i\cos\frac{\psi}{2}\sin\frac{\theta}{2}\right), \\ \gamma &= -e^{\frac{i\varphi}{2}}\left(\cos\frac{\theta}{2}\sin\frac{\psi}{2} - i\cos\frac{\psi}{2}\sin\frac{\theta}{2}\right). \end{aligned}\right\} \tag{2.91}$$

CHAPTER 3

KINEMATIC EQUATIONS

The kinematic equations relate the angular velocity vector /92
of the rigid body rotation to the time derivatives of the kinematic parameters. The form of the kinematic equations is determined depending on the kinematic parameters. Thus, the equations are different for the Euler angles, the direction cosines and the Rodriguez-Hamilton parameters. It is desirable to have a single approach to the study of the kinematic equations and, consequently, to obtaining them. In deriving the equations we will assume that the reader is familiar with such physical quantities as the angular velocity vector [9,16,29]. The derivation of the kinematic equations carried out in this chapter is based on understanding the angular velocity as the limit of the ratio of an infinitesimal rotation to the element of time. It is shown that the notation of these equations with the use of quaternions is general and recurs for all the kinematic parameters with the corresponding formalization. The general form of the kinematic equations is obtained.

The use of quaternions permits one to represent the kinematic equations in the most natural form when similar parameters characterizing the elementary rotation (angular velocity vector) and the finite rotation (the quaternion components) are related. In connection with this, the use of quaternions permits one to prove a theorem about the general solution of the kinematic equation, to construct the formal solution and to investigate a number of simple cases when one can successfully obtain a solution in elementary functions. Presented investigations are found useful for approximation methods of solutions which are

considered in the following chapter.

§ 3.1. INFINITESIMAL TRANSFORMATION. ANGULAR VELOCITY VECTOR

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The rotation transformation quaternion, for example (2.41) determines the finite rotation vector carrying the coordinate system **E** into the position **E'**. We will consider an infinitesimal transformation for which the position of the systems **E** and **E'** differ by a small quantity. This representation must correspond to a quaternion carrying out the rotation by the infinitesimal angle $\Delta\vartheta$; in accordance with the representation of the quaternion (1.11), one can see that the approximate value of this quaternion to terms of second order in $\Delta\vartheta$ equals:

$$\Delta\Lambda = 1 + \zeta\frac{\Delta\theta}{2}, \tag{3.1}$$

where $\Delta\Lambda$ is the infinitesimal transformation quaternion, ξ is the instantaneous axis of rotation, $\Delta\theta$ is the rotation angle.

Now let the quaternion (3.1) specify the rotation transformation (2.33), to which the vector **r** fixed in the coordinate system is carried into the new position **r'**.

By performing the multiplication operation, one can see that:

$$r' = \Delta\Lambda \circ r \circ \Delta\tilde{\Lambda} = \left(1 + \zeta\frac{\Delta\theta}{2}\right) \circ r \circ \left(1 - \zeta\frac{\Delta\theta}{2}\right) =$$
$$= r + \zeta\frac{\Delta\theta}{2} \circ r - r \circ \zeta\frac{\Delta\theta}{2} - \zeta \circ r \circ \zeta\frac{\Delta\theta^2}{4}.$$

Neglecting second order terms, we have the following vector increment:

$$\Delta r = r' - r = \zeta \circ r\frac{\Delta\theta}{2} - r \circ \zeta\frac{\Delta\theta}{2} = (\zeta\Delta\theta) \times r. \tag{3.2}$$

The representation of quaternion multiplication was used in formula (3.2) with the help of the vector multiplication (1.8).

By taking the ratio of both sides of the equality (3.2) to Δt and taking the limit, we obtain:

$$\frac{dr}{dt} = \left(\zeta \frac{d\Phi}{dt}\right) \times r = \omega \times r. \tag{3.3}$$

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This equality convinces us that the quantity:

$$\zeta \frac{d\Phi}{dt}$$

equals the angular velocity vector ω of the motion of the coordinate system **E'** relative to **E**. In accordance with this, the infinetimal transformation quaternion (3.1) can also be written in the form:

$$\Delta\Lambda = 1 + \frac{1}{2}\omega\Delta t, \quad d\Lambda = 1 + \frac{1}{2}\omega dt. \tag{3.4}$$

The presented result permitting one to relate the angular velocity vector to the infinitesimal rotation quaternion has an illustrative nature. Actually, we have the right to write the relation (3.4) directly on the basis of the definition of the angular velocity vector as the ratio of the infinitesimal rotation vector to the element of time dt and on the basis of the fundamental theorem 1.1 and the relation (2.51).

In the case of the direction cosines the angular velocity vector is introduced through the elements of the infinitesimal transformation matrix. The isomorphic operation of vector transformation (2.18) specifies the transformation of the matrix $\mathbf{B} = \mathbf{A}^T$, which can be represented in the form of the sum of the unit and some matrix corresponding to the small transformation:

$$\mathbf{B} = 1 + \mathbf{e}.$$

Such a matrix specifies the almost unit transformations which differ from it by an infinitesimal operator. We have, in accordance with (2.18):

$$r' = (1 + \epsilon) r = r + \epsilon r,$$

i.e.:

$$\Delta r = r' - r = \epsilon r, \tag{3.5}$$

where ε is the infinitesimal increment operator.

The inverse transformation matrix B^{-1} must be equal:

$$B^{-1} = 1 - \epsilon,$$

since in this case the relation:

$$BB^{-1} = (1 + \epsilon)(1 - \epsilon) = 1$$

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is satisfied to terms of second order in ε. But for an orthogonal matrix, the inverse matrix coincides with the transposed. Consequently,

$$B^{T} = 1 + \epsilon^{T} = 1 - \epsilon, \quad \epsilon^{T} = -\epsilon.$$

Thus, the infinitesimal transformation matrix is antisymmetric, i.e., it has the form:

$$\epsilon = \begin{vmatrix} 0 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & 0 & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & 0 \end{vmatrix} = \begin{vmatrix} 0 & \epsilon_{12} & \epsilon_{13} \\ -\epsilon_{12} & 0 & \epsilon_{23} \\ -\epsilon_{13} & -\epsilon_{23} & 0 \end{vmatrix} = \quad \epsilon_{ij} = -\epsilon_{ji},$$

$$= \begin{vmatrix} 0 & -\Delta\vartheta_3 & \Delta\vartheta_2 \\ \Delta\vartheta_3 & 0 & -\Delta\vartheta_1 \\ -\Delta\vartheta_2 & \Delta\vartheta_1 & 0 \end{vmatrix}.$$

By considering the ratio of this matrix to the element Δt and denoting the limit of this ratio as $\Delta t \to 0$ by Ω and the quantities $\frac{d\vartheta_i}{dt}$ by ω_i, we obtain:

$$\Omega = \begin{vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{vmatrix}. \tag{3.6}$$

A vector ω, whose components are the quantities $\omega_1, \omega_2, \omega_3$ corresponds to each antisymmetric matrix in three-dimensional space. The relation of the matrix elements with the vector components is established formally with the help of the Levi-Civita symbol δ_{ijk}. This symbol equals 0 is any of the indices i, j, k are the same and equal +1 or =1 depending on the evenness or oddness of the permutation of $\begin{pmatrix} 1 & 2 & 3 \\ i, & j, & k \end{pmatrix}$. The relation establishing the link between the matrix $\|\epsilon_{ij}\|$ with the vector components has the form:

$$\Delta\vartheta_i = \frac{1}{2}\sum_{j,k=1}^{3}\delta_{ijk}\epsilon_{jk}. \tag{3.7}$$

The three quantities $\Delta\vartheta_i$ (or $d\vartheta_i$) of the matrix ε are three independent parameters defining the rotation. By substituting /96 the value ε into (3.5), we obtain for the vector increment:

$$\begin{aligned} dr_1 &= r_3 d\vartheta_2 - r_2 d\vartheta_3, \\ dr_2 &= r_1 d\vartheta_3 - r_3 d\vartheta_1, \\ dr_3 &= r_2 d\vartheta_1 - r_1 d\vartheta_2, \end{aligned}$$

which can also be written in the form of the vector equality:

$$dr = d\vartheta \times r$$

or:

$$\frac{dr}{dt} = \omega \times r.$$

Since the quantity ω is defined by the vector part of the rotation transformation quaternion, then, in accordance with theorem 2.2, it is a vector. Hence, it follows that the quantities ω_i defined by the components of the quaternion $d\Lambda$ (3.4) are projections of the angular velocity vector ω (as is the quaternion) onto any coordinate system, including the basis E. Obviously, the projections on the bases E and E' are the same in this case, since $d\Lambda$ defines the transition between these bases.

We consider, in conclusion, the sequence of two infinitesimal transformations defined by the quaternions $d\Lambda_1$ and $d\Lambda_2$. The resultant transformation will equal:

$$d\Lambda = d\Lambda_2 \circ d\Lambda_1 = \left(1 + \frac{1}{2}\omega_2 dt\right) \circ \left(1 + \frac{1}{2}\omega_1 dt\right) =$$
$$= 1 + \frac{1}{2}(\omega_1 + \omega_2) dt + \frac{1}{4}\omega_2 \circ \omega_1 dt^2 \approx 1 + \frac{1}{2}(\omega_1 + \omega_2) dt,$$

i.e., the addition of the infinitesimal rotations and the corresponding angular velocities are carried out to second order. We note that the order of multiplication plays no role here; this says that the projections of the vectors $\omega_1 + \omega_2$ on neighboring coordinate systems do not differ up to second order.

It is useful to write the obtained formula separately, / 97
as a rule for multiplying the small rotation quaternions.

According to (3.4), the small rotation quaternions to second order of the rotation angle ϑ can be represented in the form:

$$\Delta\Lambda = 1 + \frac{1}{2}\theta + o(\theta),$$

where $\theta = \zeta\vartheta$ is the infinitesimal rotation vector. The formula for multiplying small transformation quaternions to terms of second order will be:

$$\Delta\Lambda_3 = \Delta\Lambda_1 \circ \Delta\Lambda_2 = \left[1 + \frac{1}{2}\theta_1 + o(\theta_1)\right] \circ \left[1 + \frac{1}{2}\theta_2 + o(\theta_2)\right] =$$
$$= 1 + \frac{1}{2}\theta_1 + \frac{1}{2}\theta_2 + o(\theta_1\theta_2) = 1 + \frac{1}{2}\theta_3 + o(\theta_3).$$

Hence, it is seen that infinitesimal rotations add as vectors.

§ 3.2. DERIVATION OF THE KINEMATIC EQUATIONS

In deriving the kinematic equations one should take into account that application in practice was found for the equations in which the projections of the angular velocity vector ω of the motion of the moving coordinate systems relative to a reference system appear either in this reference system or in the moving coordinate system (and not on any basis in general).

This explains why the projections of the quaternion onto the initial and final transformed bases, i.e., the Rodriguez-Hamilton parameters are used in the kinematic equations. We will represent two derivations of the kinematic equations.

I. We will consider reference coordinate system I and the coordinate system **E** moving with respect to it with an angular velocity ω. Let the transition from the basis I to the basis **E** be defined by the quaternion Λ with the help of a rotation operation in the form (2.44), which we write in the form:

$$\mathbf{E} = \Lambda \circ \mathbf{I} \circ \tilde{\Lambda}. \tag{3.8}$$

Because of the relative motion of the bases, the quaternion Λ will be a variable quantity. This means that the components of the quaternion λ_i $(i = 0, 1, 2, 3)$ will be functions of time $\lambda_i(t)$ and, accordingly, we will also denote the quaternion Λ as a time function $\Lambda(t)$. In order to obtain the magnitude of the rate of change of the quaternion components, we consider two neighboring positions of the coordinate system **E** at the times t and $t + dt$. Obviously, these positions will be defined relative to the reference basis I by the quaternions $\Lambda(t)$ and $\Lambda(t + dt)$. On the other hand, the relative position of the basis **E** at the times t and t + dt, will depend on the angular velocity of the motion of the basis **E** and will be specified by the infinitesimal

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motion quaternion $d\Lambda$ (Figure 3.1).

Hence, one can easily obtain the relation between the quaternions $\Lambda(t)$, $\Lambda(t+dt)$ and $d\Lambda$. However, we consider fi[illegible] of [illegible], onto which coordinate system the obtained equalities should be projected, i.e., which mappings of the given quaternions should be used. It is then necessary to proceed from the fact that it is desirable to express the quaternion $d\Lambda$ in the projections either onto the axes of the coordinate system E [then its components will be projections of the angular velocity onto the body coordinate system] or on the basis I in which the components $d\Lambda$ will be projections onto the fixed coordinate system.

In the first case, we will use theorem 2.4, according to which the eigenquaternions of each transformation are multiplied in the following manner:

$$\Lambda^*(t+dt)=\Lambda^*(t)\circ d\Lambda^*.$$

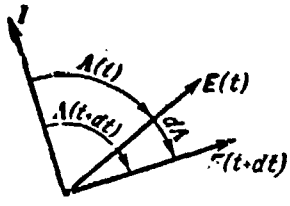

Figure 3.1.

Here the quaternions $\Lambda^*(t+dt)$ and $\Lambda^*(t)$ have the Rodriguez-Hamilton parameters as components of the transformations carrying the basis I into the bases E(t) and E (t+dt), respectively. The quaternion $d\Lambda^*$ has as its components equal magnitudes of the angular velocity projections onto the body coordinate systems E(t) and E(t+dt). By using the quaternion representation $d\Lambda$ (3.4) and denoting the mapping quaternion of the vector ω onto the basis E by ω_E, we obtain finally: /99

$$\Lambda^*(t+dt)=\Lambda^*(t)\circ\left(1+\frac{1}{2}\omega_E dt\right). \qquad (3.9)$$

Let us return to theorem 2.3, according to which the sum of the rotations $\Lambda(t)$ and $d\Lambda$ comprising the rotation $\Lambda(t+dt)$ can be projected onto the same basis. Then, in order to obtain the eigenquaternions of the transformations (the quaternions composed of the Rodriguez-Hamilton parameters) in the resulting relation, one must use only the basis I as the projection basis. For the basis E(t) and E(t+dt), either the quaternion $\Lambda(t+dt)$ or the quaternion $\Lambda(t)$ will not be the eigenquaternion of the transformation. According to theorem 2.3, we have:

$$\Lambda^*(t+dt)=\Lambda_I(t+dt)=d\Lambda_I\circ\Lambda_I(t)=d\Lambda_I\circ\Lambda^*(t),$$

i.e., the components of the quaternion $d\Lambda_I$ will be projections of the angular velocity vector onto the reference (fixed) coordinate system. By using the representation (3.4), we write the obtained the relation in the form:

$$\Lambda^*(t+dt)=\left(1+\frac{1}{2}\omega_I\,dt\right)\circ\Lambda^*(t).$$

The relations (3.9) and (3.10) are also two forms of the kinematic equation. The equations in this form have a meaning for all numerical methods of solution, since the values of the quaternion in a preceding moment of time are related to its value of the current instant of time.

However, another form for describing the kinematic equation is used more often. We consider the quantity:

$$\frac{\Lambda^*(t+dt)-\Lambda^*(t)}{dt},$$

which is a formal derivative of the quaternion as the four dimensional vector:

$$\frac{d\Lambda^*}{dt}=\frac{d\lambda_0^*}{dt}+i_1\frac{d\lambda_1^*}{dt}+i_2\frac{d\lambda_2^*}{dt}+i_3\frac{d\lambda_3^*}{dt}.$$

This derivative is taken, naturally, under the assumption of the invariant of the basis I, which is a component of the hypercomplex space H. One can see with consideration of this equality that the kinematic Equations (3.9) and (3.10) can be written respectively in the form: /100

$$\frac{d\Lambda^*}{dt} = \frac{1}{2}\Lambda^* \circ \omega_E, \tag{3.11}$$

$$\frac{d\Lambda^*}{dt} = \frac{1}{2}\omega_I \circ \Lambda^*. \tag{3.12}$$

We note that the quaternion Λ^* specifies the transition from the basis I to the basis E by the rotation operation (3.8); in accordance with theorem 2.1, the transformation of the components of an invariant angular velocity vector must be defined by the inverse rotation operation performed on the hypercomplex mappings, i.e.:

$$\omega_E = \tilde{\Lambda}^* \circ \omega_I \circ \Lambda^*. \tag{3.13}$$

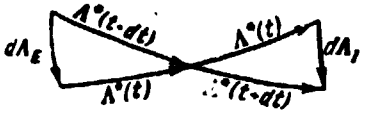

Figure 3.2.

Obviously, the relation (3.13) uniquely satisfies the Equations (3.11) and (3.12). The Equations (3.9) and (3.10) correspond to the two possible additions of quaternions on the sphere (theorem 3.2). According to this figure·

$$\Lambda^*(t+dt) = d\Lambda_I \circ \Lambda^*(t) = \Lambda^*(t) \circ d\Lambda_E,$$

and the relation of the arcs $d\Lambda_I$ and $d\Lambda_E$ are determined by the transition operation:

$$d\Lambda_E = \tilde{\Lambda}^* \circ d\Lambda_I \circ \Lambda^*,$$

which is an expression of the equality (3.13).

II. The kinematic equations can be obtained in another way. Let the transition from the reference basis I to the basis E be defined by the quaternion Λ (3.8). Then, for any basis vector, as in general for any vector fixed in the basis **E**, the transformation is defined by the same operation:

$$r' = \Lambda^* \circ r \circ \tilde{\Lambda}^*.$$

By considering the change of r' in the reference basis, i.e., the motion occurring only because of the change of Λ^*, we obtain the magnitude of the local derivative of the radius vector: /101

$$\frac{dr'}{dt} = \frac{d\Lambda^*}{dt} \circ r \circ \tilde{\Lambda}^* + \Lambda^* \circ r \circ \frac{d\tilde{\Lambda}^*}{dt} = \\ = \frac{d\Lambda^*}{dt} \circ \tilde{\Lambda}^* \circ r' - r' \circ \frac{d\Lambda^*}{dt} \circ \tilde{\Lambda}^*, \qquad (3.14)$$

since $r \circ \tilde{\Lambda}^* = \tilde{\Lambda}^* \circ r'$, and $\Lambda^* \circ r = r' \circ \Lambda^*$, and from the equality $\Lambda^* \circ \tilde{\Lambda}^* = 1$, it follows that:

$$\frac{d\Lambda^*}{dt} \circ \tilde{\Lambda}^* + \Lambda^* \circ \frac{d\tilde{\Lambda}^*}{dt} = 0,$$

i.e.:

$$\frac{d\tilde{\Lambda}^*}{dt} = -\tilde{\Lambda}^* \circ \frac{d\Lambda^*}{dt} \circ \tilde{\Lambda}^*, \qquad (3.15)$$

and the magnitude of the quaternion derivative is understood in the sense of (3.11), i.e., it is the local change of the quaternion relative to the reference basis. On the other hand, the local derivative of a vector fixed in the basis **E** is defined by the relation (3.3). In order to obtain the relation of the quantity $\frac{d\Lambda^*}{dt}$ to the angular velocity vector ω, it is necessary to represent the vector equality (3.3) in the form of quaternion products:

$$\omega \times r' = \frac{1}{2}(\omega_1 \circ r' - r' \circ \omega_1), \qquad (3.16)$$

which follows from the representation (1.8), where

$$\omega_1 = \Omega_1 i_1 + \Omega_2 i_2 + \Omega_3 i_3 \qquad (3.17)$$

is a quaternion composed of the projections of the vector ω onto the axes of the reference basis, i.e., a mapping of ω onto I.

By comparing the relations (3.14) and (3.16), we obtain at once:

$$\omega_I = 2\frac{d\Lambda^*}{dt}\circ\tilde{\Lambda}^*, \tag{3.18}$$

from which the form of the kinematic Equation (3.12) follows: The (3.11) is obtained from (3.18) by taking into account the fact that the mapping quaternion ω_E is obtained from ω_I by the operation (3.13) / 102

From the relation (3.11), it follows that

$$\omega_E = 2\tilde{\Lambda}^*\circ\frac{d\Lambda^*}{dt}, \tag{3.19}$$

and the magnitude of this quaternion:

$$\omega_E = \omega_1 i_1 + \omega_2 i_2 + \omega_3 i_3 \tag{3.20}$$

is determined by the projections of the vector ω onto the body coordinate system E.

By using the representations (3.17) and (3.20), we write the equations defining the angular velocity vector components through the derivatives of the transformation quaternion (3.18) and (3.19) in the form of scalar relations (here and subsequently, we will omit the asterisk with the quaternion components, understanding that only the Rodriguez-Hamilton parameters appear in the kinematic equations):

$$\left.\begin{aligned}\Omega_1 &= 2(\lambda_0\dot{\lambda}_1 - \dot{\lambda}_0\lambda_1 + \lambda_2\dot{\lambda}_3 - \dot{\lambda}_2\lambda_3),\\ \Omega_2 &= 2(\lambda_0\dot{\lambda}_2 - \dot{\lambda}_0\lambda_2 + \lambda_3\dot{\lambda}_1 - \lambda_1\dot{\lambda}_3),\\ \Omega_3 &= 2(\lambda_0\dot{\lambda}_3 - \dot{\lambda}_0\lambda_3 + \lambda_1\dot{\lambda}_2 - \lambda_2\dot{\lambda}_1);\end{aligned}\right\} \qquad (3.21)$$

$$\left.\begin{aligned}\omega_1 &= 2(\dot{\lambda}_1\lambda_0 - \dot{\lambda}_0\lambda_1 + \lambda_3\dot{\lambda}_2 - \lambda_2\dot{\lambda}_3),\\ \omega_2 &= 2(\dot{\lambda}_2\lambda_0 - \dot{\lambda}_0\lambda_2 + \lambda_1\dot{\lambda}_3 - \lambda_3\dot{\lambda}_1),\\ \omega_3 &= 2(\dot{\lambda}_3\lambda_0 - \dot{\lambda}_0\lambda_3 + \lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2).\end{aligned}\right\} \qquad (3.22)$$

The Equations (3.21) and (3.22) define the magnitude of the angular velocity, which is a physical characteristic of the motion and can be measured. One can snow that the determinants of the coefficients of the velocities (i.e., of the derivatives $\dot{\lambda}_i$ in the systems (3.21) and (3.22) always equals 1. This means, firstly, that the inversion of these systems is always possible and, secondly, that the kinematic equations in the form (3.11) (3.12) are not defenerate any values of the parameters. This property is very important for applications, in particular for numerical solution of the equations.

We write Equations (3.11) and (3.12) in the form of the scalar relations: /103

$$\left.\begin{aligned}2\dot{\lambda}_0 &= -\omega_1\lambda_1 - \omega_2\lambda_2 - \omega_3\lambda_3,\\ 2\dot{\lambda}_1 &= \omega_1\lambda_0 + \omega_3\lambda_2 - \omega_2\lambda_3,\\ 2\dot{\lambda}_2 &= \omega_2\lambda_0 + \omega_1\lambda_3 - \omega_3\lambda_1,\\ 2\dot{\lambda}_3 &= \omega_3\lambda_0 + \omega_2\lambda_1 - \omega_1\lambda_2;\end{aligned}\right\} \qquad (3.23)$$

$$\left.\begin{aligned}2\dot{\lambda}_0 &= -\Omega_1\lambda_1 - \Omega_2\lambda_2 - \Omega_3\lambda_3,\\ 2\dot{\lambda}_1 &= \Omega_1\lambda_0 + \Omega_2\lambda_3 - \Omega_3\lambda_2,\\ 2\dot{\lambda}_2 &= \Omega_2\lambda_0 + \Omega_3\lambda_1 - \Omega_1\lambda_3,\\ 2\dot{\lambda}_3 &= \Omega_3\lambda_0 + \Omega_1\lambda_2 - \Omega_2\lambda_1.\end{aligned}\right\} \qquad (3.24)$$

We note once again that the kinematic equations describe the motion of one coordinate system relative to another. Thus, the relative angular velocity vector appears in them.

We will consider in conclusion how the relative motions are added.

Let the quaternion Λ^* specify a transition from the coordinate system I to the system E, and the quaternion M^* — the transition from the coordinate system E to the system E'. Let the angular velocity of the coordinate system E relative to the system I equal ω_1, and the angular velocity of the coordinate system E' relative to the system E equal ω_2. The kinematic equations describing the motion of the coordinate systems, according to (3.11) and (3.12), have form:

$$\dot{\Lambda}^* = \frac{1}{2}\omega_{1I} \circ \Lambda^* = \frac{1}{2}\Lambda^* \circ \omega_{1E}. \tag{3.25}$$

Analogous relations also determine the relative motion of the basis E' relative to the basis E:

$$\dot{M}^* = \frac{1}{2}\omega_{2E} \circ M^* = \frac{1}{2}M^* \circ \omega_{2E'}, \tag{3.26}$$

where ω_{2E} and $\omega_{2E'}$ are quaternions composed of the projections of the vector ω_2 onto the reference (E) and body (E') bases.

We consider the resulting quaternions N^*; as follows from theorem 2.4, for the eigenquaternion consisting of the Rodriguez-Hamilton parameters, the relation: /104

$$N^* = \Lambda^* \circ M^*.$$

is valid. Differentiating this equality, we obtain:

$$\frac{dN^*}{dt} = \frac{d\Lambda^*}{dt} \circ M^* + \Lambda^* \circ \frac{dM^*}{dt}.$$

We substitute into it the values of the derivatives of the quaternion components in accordance with the first equalities of (3.25) and (3.26):

$$\frac{dN^*}{dt} = \frac{1}{2}\omega_{1I} \circ \Lambda^* \circ M^* + \frac{1}{2}\Lambda^* \circ \omega_{2E} \circ M^* =$$

$$= \frac{1}{2}\omega_{1I} \circ N^* + \frac{1}{2}\Lambda^* \circ \omega_{2E} \circ \tilde{\Lambda}^* \circ N^*.$$

Since the projections of the vector ω_2 on the basis of E are related to its projections on the basis I are an operation of the form (3.13), then the mapping of the vector ω_2 onto the basis I will equal:

$$\omega_{2I} = \Lambda^{\circ} \circ \omega_{2E} \circ \tilde{\Lambda}^{\circ}.$$

By substituting this relation in the equality obtained above, we have:

$$\frac{dN^{\circ}}{dt} = \frac{1}{2}(\omega_{1I} + \omega_{2I}) \circ N^{\circ}, \tag{3.27}$$

i.e., the kinematic equation for the combined motion has the form (3.12). Since the mappings of the vectors ω_1 and ω_2 are represented onto the same (reference) basis, the addition of the angular velocity vectors results. The equality (3.27) is a reflection of the known kinematic law of angular velocity addition for the addition of relative motions.

By analogy, by considering the second equalities of (3.25) and (3.26), we obtain:

$$\frac{dN^{\circ}}{dt} = \frac{1}{2}\Lambda^{\circ} \circ \omega_{1E} \circ M^{\circ} + \frac{1}{2}\Lambda^{\circ} \circ M^{\circ} \circ \omega_{2E'} =$$
$$= \frac{1}{2}N^{\circ} \circ \tilde{M}^{\circ} \circ \omega_{1E} \circ M^{\circ} + \frac{1}{2}N^{\circ} \circ \omega_{2E'}.$$

The projections of the vector ω_1 onto the basis E' are determined /105 by the rotation operation, the inverse operation of the basis transformation:

$$\omega_{1E'} = \tilde{M}^{\circ} \circ \omega_{1E} \circ M^{\circ},$$

Hence, it follows that:

$$\frac{dN^{\circ}}{dt} = \frac{1}{2}N^{\circ} \circ (\omega_{1E'} + \omega_{2E'}), \tag{3.28}$$

i.e., the equation takes the form (3.11) and the addition of the angular velocity vectors also results. We note that the relations (3.27) and (3.28) could have been obtained directly from (3.11) and (3.12) by taking into account that the velocity of the coordinate system E' relative to the system I equals $\omega_1 + \omega_2$. The obtained conclusion illustrates the validity of the kinematic relations.

§ 3.3. KINEMATIC EQUATIONS IN OTHER PARAMETERS

I. Direction cosines. For the derivation of the kinematic equations relating the derivaties of the cosines with the angular velocity vector components, we consider the relation (2.18) giving the transformation of a vector onto a vector for an orthogonal transformation. This relation corresponds to the matrix equality (2.17). We write these equalities by substituting for the matrix $\mathbf{A}^T$ the equal matrix **B**.

$$\mathbf{r}' = \mathbf{B}\mathbf{r}, \quad \mathbf{R}' = \mathbf{B}\mathbf{R}. \tag{3.29}$$

This transformation desribes the change of a vector fixed in the moving body coordinate system, which occurs because of the motion of the basis. The components of **r** in the reference **I** and final **E** bases are the same. Hence, it follows that the matrix **R** is constant.

By considering the change of the vector occurring because of the motion of the basis, we have for the matrix relation (3.29):

$$\dot{\mathbf{R}}' = \dot{\mathbf{B}}\mathbf{R} = \dot{\mathbf{B}}\mathbf{B}^T\mathbf{R}'. \tag{3.30}$$

On the other hand, the change of the vector components for an infinitesimal transformation (3.5) was considered in paragraph 3.1.

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We will find the dependence of the quantity $\dot{R}'$ on the angular velocity. Let us have two neighboring values $R'(t+dt)$ and $R'(t)$, which differ by an infinitesimal operator (3.5):

$$R'(t+\Delta t)=(1+\epsilon)R'(t).$$

From this relation, we have:

$$\Delta R'=R'(t+\Delta t)-R'(t)=\epsilon R'(t);$$

By taking the ratio of this quality with Δt and taking the limit, we obtain:

$$\dot{R}'=(d\epsilon/dt)R'=\Omega R', \tag{3.31}$$

where $d\epsilon/dt$ is the angular velocity matrix (3.6).

We note that the equality (3.31) is the matrix form of the vector relation (3.3). In addition, since the relation (3.29) describes the motion of the vector r relative to the reference basis, the elements of the matrices R and R', as well as the angular velocity matrix Ω, are respectively projections of the vectors r, r' and ω onto the fixed coordinate system I. In connection with this, we will subsequently denote the matrix Ω in equality (3.31) by the index of the basis. Since the projections of the vector ω onto the basis I are denoted by the quantities $\Omega_i(\omega=\Omega_1 i_1+\Omega_2 i_2+\Omega_3 i_3)$, then:

$$\Omega_I=\begin{vmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{vmatrix}.$$

By comparing the relations (3.31) and (3.30), we obtain

$$\Omega_I = \dot{B}B^T. \tag{3.32}$$

By multiplying the expression (3.32) on the right by the nonzero matrix B, we have $(BB^T = B^TB = 1)$:

$$\dot{B} = \Omega_I B. \tag{3.33}$$

This is the first form of the kinematic equation for the direction cosines, in which the magnitudes of the projections the angular velocity vector are taken on the fixed basis. /107

To obtain the second form of the equation, we consider the matrix Ω_I, which obviously is an operator matrix. It was shown in paragraph 2.1 that with a basis transformation the matrix operator satisfies the similar transformation (2.27). Since $A = B^T$, then, in accordance with (2.27), the operator Ω in the basis E (we denote it by Ω_E) will be related to Ω_I by the following similar transformation:

$$\Omega_E = B^T \Omega_I B. \tag{3.34}$$

The relation (3.34) permits one to find the projection of the angular velocity vector ω onto the basis E. These components are determined by the following operator:

$$\Omega_E = \begin{vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{vmatrix},$$

where $\omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$, and ω_i is the projection of the vector ω onto the basis E.

By substituting in (3.33) the angular velocity matrix Ω_I expressed by Ω_E in accordance with (3.34), we obtain the second form of the kinematic equation, in which the magnitudes of the projections of the angular velocity are taken on the body coordinate system:

$$\dot{B} = B\Omega_E. \tag{3.35}$$

We note at once that the form of the kinematic Equations (3.33) and (3.35) is similar to the form of the Equations (3.12) and (3.11), respectively.

In the case when the orthogonal transformation is specified in the form where the matrix **A** is used, the kinematic equations for the matrix **A** are obtained from (3.33) and (3.35) taking into account that:

$$AB = BA = 1, \quad \dot{A}B + A\dot{B} = 0, \quad \dot{B} = -B\dot{A}B.$$

We have:

$$\dot{A} = -\Omega_E A = -A\Omega_I. \tag{3.36}$$

By analogy to relation (3.32) proceeding from the equality (3.35), we obtain the matrix defined by the projections of the angular vector onto the body axes as a function of the direction cosines and their derivatives: /108

$$\Omega_E = B^T\dot{B} = -\dot{A}A^T. \tag{3.37}$$

Since the determinant of the matrix of an orthogonal transformation equals one, then the inversion of the Equations (3.32) and (3.37) is always possible. Thus, the kinematic Equations (3.33), (3.35) and (3.35) are not degenerate for any values of the parameters, i.e., for any position of the rigid body.

We will write the kinematic equations in scalar form. Since in writing the orthogonal transformations the matrix **A** is more frequently used, we write the equations for the elements a_{ij} keeping in mind that $b_{ij} = a_{ji}$. By using equation (3.36), we write the following eighteen scalar equations being the Poisson equations:

$$\left.\begin{aligned} \dot{a}_{1i} &= -\omega_2 a_{3i} + \omega_3 a_{2i}, \quad \dot{a}_{k1} = -\Omega_3 a_{k2} + \Omega_2 a_{k3}, \\ \dot{a}_{2i} &= -\omega_3 a_{1i} + \omega_1 a_{3i}, \quad \dot{a}_{k2} = -\Omega_1 a_{k3} + \Omega_3 a_{k1}, \\ \dot{a}_{3i} &= -\omega_1 a_{2i} + \omega_2 a_{1i}, \quad \dot{a}_{k3} = -\Omega_2 a_{k1} + \Omega_1 a_{k2} \\ &(i, k = 1, 2, 3). \end{aligned}\right\} \tag{3.38}$$

The relations (3.32) and (3.37) determine the magnitudes of the angular velocity components:

$$\Omega_E = -\dot{A}A^T \quad \text{and} \quad \Omega_I = A^T\dot{A}.$$

We have:

$$\left.\begin{aligned} \omega_1 &= -\dot{a}_{31}a_{21} - \dot{a}_{32}a_{22} - \dot{a}_{33}a_{23} = \dot{a}_{21}a_{31} + \dot{a}_{22}a_{32} + \dot{a}_{23}a_{33}, \\ \omega_2 &= -\dot{a}_{11}a_{31} - \dot{a}_{12}a_{32} - \dot{a}_{13}a_{33} = \dot{a}_{31}a_{11} + \dot{a}_{32}a_{12} + \dot{a}_{33}a_{13}, \\ \omega_3 &= -\dot{a}_{21}a_{11} - \dot{a}_{22}a_{12} - \dot{a}_{23}a_{13} = \dot{a}_{11}a_{21} + \dot{a}_{12}a_{22} + \dot{a}_{13}a_{23}; \\ \Omega_1 &= -\dot{a}_{12}a_{13} - \dot{a}_{22}a_{23} - \dot{a}_{32}a_{33} = \dot{a}_{13}a_{12} + \dot{a}_{23}a_{22} + \dot{a}_{33}a_{32}, \\ \Omega_2 &= -\dot{a}_{13}a_{11} - \dot{a}_{23}a_{21} - \dot{a}_{33}a_{31} = \dot{a}_{11}a_{13} + \dot{a}_{21}a_{23} + \dot{a}_{31}a_{33}, \\ \Omega_3 &= -\dot{a}_{12}a_{11} - \dot{a}_{22}a_{21} - \dot{a}_{32}a_{31} = \dot{a}_{12}a_{11} + \dot{a}_{22}a_{21} + \dot{a}_{32}a_{31}. \end{aligned}\right\} \tag{3.39}$$

Nine of the kinematic Poisson Equations (3.38) which contain the projection of the angular velocity vector onto the body or onto the fixed axes, describe the motion of the three basis vectors of the body basis. For the orientation of the body it is obviously /109 sufficient to specify the position of only two of its axes. Thus, any triplet of equations can be written. One triplet of equations describes the motion of one axis; there being one coupling equation (an equation fixing the length of the basis vector, i.e., the equation of the norm). In addition, when the motion of one axis is specified (the first triplet of equations), there are two relations imposed on the second triplet of equations, which fix

the length of the basis vector and its perpendicularity to the first. Thus, the system of kinematic equations contains three independent parameters.

The Equations (3.33), (3.35) and (3.36) have the most general form of kinematic equations coinciding with the form of the equations in quaternions (3.11), (3.12). The multiplication of the transformation quaternion and the quaternion mapping the angular velocity vector onto some basis or another was carried out in the case of the quaternion equations. In the case of the same equations for the direction cosines, multiplication of the transformation matrix by the angular velocity matrix is carried out. We note that one can obtain from Equation (3.33), (3.35) and (3.36) equations from any kinematic parameters only if the orthogonal transformation is expressed in them (which is always possible).

We will show in conclusion that Equations (3.33), (3.35) and (3.36) satisfy the well-known law of kinematics, according to which the vector addition of angular velocities is performed for relative motions. We consider two transformations: transition from the basis **I** to the basis **E** defined by the matrix A_1, and the transition from the basis **E** to the basis **E'** specified by the matrix A_2. The resultant transformation will be defined by the matrix $A_3 = A_2 A_1$. Let ω_1 be the angular velocity of the basis **E** relative to the basis **I**, and ω_2 be the rotational velocity of the basis **E'** relative to the basis **E**. Further, let Ω_1 and Ω_2 be the corresponding matrices of these velocities. In accordance with (3.36), we have two systems of kinematic equations defining the relative motion of the bases **I**, **E**, **E'**:

$$\dot{A}_1 = -\Omega_{1E}A_1 = -A_1\Omega_{1I}, \quad \dot{A}_2 = -\Omega_{2E}A_2 = -A_2\Omega_{2E}.$$

By differentiating the matrix A_3 of the resultant transformation, we obtain: / 110

$$\dot{A}_3 = \dot{A}_2 A_1 + A_2 \dot{A}_1 = -\Omega_{2E'} A_2 A_1 - A_2 \Omega_{1E} A_1 = \\ = -A_2 \Omega_{2E} A_1 - A_2 A_1 \Omega_{1I}. \qquad (3.40)$$

matrix Ω_{1E} is an operator "projecting" onto the basis E; for the transition to the basis E' it experiences the similar transformation:

$$\Omega_{1E'} = A_2 \Omega_{1E} A_2^T.$$

The kinematic equation:

$$\dot{A}_3 = -(\Omega_{1E'} + \Omega_{2E'}) A_3,$$

which is the first form of the kinematic equation for the motion with angular velocity $\omega_1 + \omega_2$, follows from the equalities (3.40) with consideration of this.

If the operator Ω_{2E} is expressed in the basis, I then it will have the form

$$\Omega_{2I} = A_1^T \Omega_{2E} A_1,$$

thanks to which we obtain from the equalities (3.40) the second form of the kinematic equation:

$$\dot{A}_3 = -A_3 (\Omega_{1I} + \Omega_{2I}).$$

The addition of the angular velocity components also results in thie equation.

II. Euler and Krylov Angles. The kinematic equations in the parameters, which are the Euler or Krylov angles, are well known [9,16,31]. Thus, we will discuss them very briefly. The relation of the angular velocity components with the derivatives of the Euler or Krylov angles can be obtained from the relations (3.32) and (3.37) by using the value of the matrix (2.67) or (2.70), respectively. However, it is easier to use the fact that the angular velocity vector equals the sum of the angular velocity

vectors of the relative motion components. Since the transformation is specified by three successive plane rotations by the angles φ, ϑ and ψ, then the angular velocity of each rotation is directed along the axis of rotation and equals $\dot{\varphi}$, $\dot{\vartheta}$ and $\dot{\psi}$. We have from Figure 2.1 for the Euler angles: /111

$$\left.\begin{aligned} \Omega_1 &= \dot{\psi}\cos\varphi + \dot{\vartheta}\sin\psi\sin\varphi, & \omega_1 &= \dot{\psi}\cos\vartheta + \dot{\varphi}\sin\vartheta\sin\psi, \\ \Omega_2 &= \dot{\psi}\sin\varphi - \dot{\vartheta}\sin\psi\cos\varphi, & \omega_2 &= -\dot{\psi}\sin\vartheta + \dot{\varphi}\cos\vartheta\sin\psi, \\ \Omega_3 &= \dot{\varphi} + \dot{\vartheta}\cos\psi; & \omega_3 &= \dot{\vartheta} + \dot{\varphi}\cos\psi. \end{aligned}\right\} \qquad (3.41)$$

We obtain the following relations for Krylov angles in accordance to Figure 2.2:

$$\left.\begin{aligned} \Omega_1 &= \dot{\vartheta}\cos\varphi\cos\psi - \dot{\psi}\sin\varphi, & \omega_1 &= \dot{\vartheta} - \dot{\varphi}\sin\psi, \\ \Omega_2 &= \dot{\vartheta}\cos\psi\sin\varphi + \dot{\psi}\cos\varphi, & \omega_2 &= \dot{\varphi}\cos\psi\sin\vartheta + \dot{\psi}\cos\vartheta, \\ \Omega_3 &= \dot{\varphi} - \dot{\vartheta}\sin\psi; & \omega_3 &= \dot{\varphi}\cos\psi\cos\vartheta - \dot{\psi}\sin\vartheta, \end{aligned}\right\} \qquad (3.42)$$

where, as in (3.41), Ω_k and ω_k $(k = 1, 2, 3)$ are the projections of the angular velocity vector onto the fixed (I) and body (E) bases.

The kinematic equations are obtained by inverting the Equations (2.41) and (2.42). The determinant of the system of Equations (3.41) resolved relative to $\dot{\varphi}$, $\dot{\psi}$ and $\dot{\vartheta}$ equals $\sin\psi$ and thus, the system (3.41) is not invertible for $\psi = k\pi$ $(k = 0, 1, \ldots)$. These values correspond to the loss of one degree of freedom (coincidence of the axes of rotation φ and ϑ). We have for $\psi \neq k\pi$:

$$\left.\begin{aligned} \dot{\psi} &= \Omega_1\cos\varphi + \Omega_2\sin\varphi, & \dot{\psi} &= \omega_1\cos\vartheta - \omega_2\sin\vartheta, \\ \dot{\vartheta} &= \Omega_1\frac{\sin\varphi}{\sin\psi} - \Omega_2\frac{\cos\varphi}{\sin\psi}, & \dot{\vartheta} &= \omega_3 - \omega_1\sin\vartheta\operatorname{ctg}\psi - \\ & & & \quad - \omega_2\cos\vartheta\operatorname{ctg}\psi, \\ \dot{\varphi} &= \Omega_3 - \Omega_1\sin\varphi\operatorname{ctg}\psi + & \dot{\varphi} &= \omega_1\frac{\sin\vartheta}{\sin\psi} + \omega_2\frac{\cos\vartheta}{\sin\psi}. \\ & \quad + \Omega_2\cos\varphi\operatorname{ctg}\psi, & & \end{aligned}\right\} \qquad (3.43)$$

As is seen from these equations, the rate of change of the angles φ and ϑ tends to infinity with the approach to the point $\psi = k\pi$.

For the solution of the Equations (3.42) relative to the quantities $\dot{\varphi}$, $\dot{\psi}$ and $\dot{\vartheta}$, it is necessary that the determinant of each of the systems equaling $\cos\psi$ not be equal to zero. Coinci- /112 dence of the directions of $\dot{\varphi}$ and $\dot{\vartheta}$ occurs at the points $\psi=(2k+1)\frac{\pi}{2}$, and the system of such parameters looses one degree of freedom. We have, in accordance with (3.42), at the points $\psi\neq(2k+1)\frac{\pi}{2}$:

$$\left.\begin{aligned}
&\dot{\psi}=\Omega_2\cos\varphi-\Omega_1\sin\varphi, && \dot{\psi}=\omega_2\cos\vartheta-\omega_3\sin\vartheta,\\
&\dot{\vartheta}=\Omega_2\frac{\sin\varphi}{\cos\psi}+\Omega_1\frac{\cos\varphi}{\cos\psi}. && \dot{\vartheta}=\omega_1+\omega_2\sin\vartheta\operatorname{tg}\psi+\omega_3\cos\vartheta\operatorname{tg}\psi,\\
&\dot{\varphi}=\Omega_3+\Omega_1\cos\varphi\operatorname{tg}\psi+ && \dot{\varphi}=\omega_3\frac{\cos\vartheta}{\cos\psi}+\omega_2\frac{\sin\vartheta}{\cos\psi}.\\
&\qquad+\Omega_2\sin\varphi\operatorname{tg}\psi, &&
\end{aligned}\right\}\qquad(3.44)$$

As in the preceding case, the rates of change of the angles φ and ϑ tend to infinity with the approach to the point $\psi=(2k+1)\frac{\pi}{2}$, although Ω_i and ω_i remain finite.

We noted above that the equalities (3.41) and (3.42) can be obtained from the general relations (3.32) and (3.37). However, the kinematic Equations (3.43) and (3.44) do not correspond to the general Equations (3.33) and (3.35) in which the derivatives of the direction cosines, which are combinations of the angles φ, ϑ and ψ and their derivatives, appear.

Thus, the kinematic Equations (3.33) and (3.35) are not degenerate for any rigid body motion, which we do not have in the case of the kinematic equations in the Euler and Krylov angles.

III. Cayley-Klein Parameters. The method developed in paragraph 3.2 for deriving the kinematic equations can be applied (as in the case of quaternions and the direction cosines) for the Cayley-Klein parameters. We find first the Cayley-Klein parameters for an infinitesimal transformation, i.e., we determine their dependence on the infinitesimal rotation vector and, consequently, on the angular velocity. For this, it is most convenient to use the representation of the unitary matrix of the transformation

in the form of its resolution in the Pauli spin matrices (2.87), i.e., to represent the Cayley-Klein in the form: /113

$$\left.\begin{aligned} \alpha &= \cos\frac{\Phi}{2} + i\gamma_3 \sin\frac{\Phi}{2}, \quad \delta = \cos\frac{\Phi}{2} - i\gamma_3 \sin\frac{\Phi}{2}, \\ \beta &= (\gamma_2 + i\gamma_1)\sin\frac{\Phi}{2}, \quad \gamma = -(\gamma_2 - i\gamma_1)\sin\frac{\Phi}{2}. \end{aligned}\right\} \qquad (3.45)$$

This relation is obtained by comparing the two representations of the transformation matrix in the form (2.86) and (2.87). They permit one to express the Cayley-Klein parameters as functions o the rotation angle and the direction cosines of the unit transformation vector. We note that the relations (3.45) correspond to the equalities (2.89) relating the Cayley-Klein parameters to the components of a quaternion.

In accordance with (3.45), the infinitesimal transformation, i.e., the transformation with the small rotation angle $\Delta\Phi$, will correspond to the matrix:

$$\Delta U = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \begin{vmatrix} 1 + i\gamma_3 \frac{\Delta\Phi}{2} & (\gamma_2 + i\gamma_1)\frac{\Delta\Phi}{2} \\ -(\gamma_2 - i\gamma_1)\frac{\Delta\Phi}{2} & 1 - i\gamma_3 \frac{\Delta\Phi}{2} \end{vmatrix},$$

obtained, obviously, with consideration of quantities to first order relative to $\Delta\Phi$ in the formulae (3.45). We represent this matrix defining an almost identical transformation in the form of the sum of the unit matrix and the infinitesimal rotation matrix:

$$\Delta U = 1 + U_\Delta = 1 + \begin{vmatrix} i\gamma_3 \frac{\Delta\Phi}{2} & (\gamma_2 + i\gamma_1)\frac{\Delta\Phi}{2} \\ -(\gamma_2 - i\gamma_1)\frac{\Delta\Phi}{2} & -i\gamma_3 \frac{\Delta\Phi}{2} \end{vmatrix}. \qquad (3.46)$$

We consider the ratio of the matrix U_Δ to the element of time Δt and take the limit of this ratio for $\Delta t \to 0$. In this case the quantities $\nu \cdot \frac{\Delta\Phi}{\Delta t}$ become the projections of the angular

velocity vector. By denoting the matrix of the ratio $U_\Delta/\Delta t$ by $\mathbf{V}$, we obtain

$$\mathbf{V}=\frac{1}{2}\begin{vmatrix} i\omega_3 & \omega_2+i\omega_1 \\ -\omega_2+i\omega_1 & -i\omega_3 \end{vmatrix}. \qquad (3.47)$$

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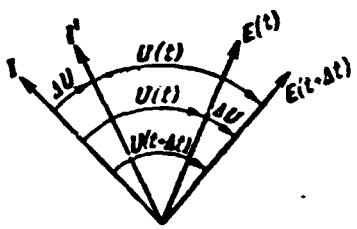

Figure 3.3.

By using the resolution of the matrix in the Pauli spin matrices (2.84), one can see that:

$$\mathbf{V}=\frac{i}{2}(\omega_1\mathbf{p}_1+\omega_2\mathbf{p}_2+\omega_3\mathbf{p}_3), \qquad (3.48)$$

i.e., the angular velocity matrix $\mathbf{V}$ is resolved like a vector along the "basis vectors" which are the spin matrices. Here, of course, there appears the complete analogy of quaternions and the spin matrices. Hence, it follows that the infinitesimal transformation matrix (3.46) is representable with consideration of (3.47) in the following form:

$$\Delta\mathbf{U}=\mathbf{1}+\mathbf{V}\Delta t. \qquad (3.49)$$

Now we can turn to the formulation of the kinematic equations. Let the position of the basis vectors of the basis $\mathbf{E}$ relative to the reference basis I be determined by a transformation of the type (2.80) specified by the unitary matrix $\mathbf{U}$. Let us consider two neighboring positions of the basis $\mathbf{E}$ defined by the values $\mathbf{U}(t)$ and $\mathbf{U}(t+\Delta t)$. Obviously, the transition from the basis I to the basis $\mathbf{E}$ at the time $t+\Delta t$ specified by the matrix $\mathbf{U}(t+\Delta t)$ can be represented in the form of the sum of transformations: the transformation $\mathbf{U}(t)$ and the infinitesimal transformation $\Delta\mathbf{U}$ (Figure 3.3).

In accordance with the rules for multiplying operators, we have with the addition of the transformations:

$$U(t+\Delta t)=\Delta U U(t).$$

One should note that since the matrix ΔU specifies a transition between two positions of the basis E then the angular velocity matrix V contains the components of the angular velocity vector projection onto the body axes. In accordance with this, in the obtained expression the infinitesimal rotation matrix (3.49) will include in its components the projection of the angular velocity vector ω onto the axes of the basis E, i.e., the quantities which we have denoted by ω_i. We replace the matrix / 115
ΔU in the obtained expression according to (3.49), and also, in accordance with this discussion, we will denote the matrix V with the basis index:

$$U(t+\Delta t)=U(t)+V_E U(t)\Delta t.$$

We denote:

$$\lim_{\Delta t\to 0}\frac{U(t+\Delta t)-U(t)}{\Delta t}=\dot{U}(t)$$

and, in accordance with the rules of matrix differentiation, we obtain:

$$\dot{U}=\begin{vmatrix}\dot{\alpha} & \dot{\beta}\\ \dot{\gamma} & \dot{\delta}\end{vmatrix}.$$

Taking the limit as $\Delta t\to 0$, we obtain the first form of the kinematic equations for the Cayley-Klein parameters:

$$\dot{U}=V_E U. \tag{3.50}$$

From this equation we obtain at once the magnitudes of the angular velocity vector projections onto the body axes by multiplying on the right by the Hermitian conjugate of the matrix U:

$$V_E=\dot{U}U^{+}. \tag{3.51}$$

We note at once the formal similarity of the relations (3.50) and (3.51) to the analogous Equations (3.35) and (3.37), and also the similarity of (3.50) to (3.11). The different order of multiplication in the formulae (3.36), (3.50) and (3.11) is explained by the order of multiplying the operators of the successive transformations under consideration and the eigenquaternions (cf. theorem 2.4). We will write these equations in the coordinates. We have from Equations (3.50):

$$\begin{vmatrix} \dot{\alpha} & \dot{\beta} \\ \dot{\gamma} & \dot{\delta} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} i\omega_3 & \omega_2 + i\omega_1 \\ -\omega_2 + i\omega_1 & -i\omega_3 \end{vmatrix} \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}.$$

By performing the matrix multiplication, we obtain

$$\left.\begin{aligned} \dot{\alpha} &= \frac{i\omega_3}{2}\alpha + \frac{\omega_2 + i\omega_1}{2}\gamma, \quad \dot{\beta} = \frac{i\omega_3}{2}\beta + \frac{\omega_2 + i\omega_1}{2}\delta, \\ \dot{\gamma} &= -\frac{i\omega_3}{2}\gamma + \frac{i\omega_1 - \omega_2}{2}\alpha, \quad \dot{\delta} = -\frac{i\omega_3}{2}\delta + \frac{i\omega_1 - \omega_2}{2}\beta. \end{aligned}\right\} \qquad (3.52)$$

We note that of the four presented equations the second pair /116 (for β and δ) is a consequence of the first pair for α and γ and is obtained from it with consideration of the relation (2.78) between the Cayley-Klein parameters (it is the complex conjugate of the first pair of equations). If the expressions for α and γ are replaced by the elements of the quaternions (2.89), then one can verify that the Equations (3.52) become the Equations (3.23) for quaternions. It should be noted that since in Equations (3.52) α and γ (and consequently, β and δ) are complex quantities. Then, the inherently scalar real equations corresponding to the Equation (3.50) will be the Equations (3.23) for the quaternion; the coupling Equations (2.78) becoming the quaternion norm equation.

In order to obtain the second form of the equations containing the projection of the angular velocity vector onto the axes of the fixed basis, let us consider another sequence of

transformations. Let the position of the coordinate system E at the time $t+\Delta t$ be defined, as previously, by the unitary matrix $U(t+\Delta t)$. In contrast to the preceding case, this position can be represented in the form of an infinitesimal transformation: a transition from the basis I to the basis I' such that the next transition from I' to E at the time $t+\Delta t$ is defined by the matrix U(t).

Thus, if the matrix $U(t+\Delta t)$ in the preceding case was represented in the form of the sequence of matrices U(t) and ΔU (cf. Figure 3.3), then in this version the sequence is inverted. The matrix ΔU still determines the infinitesimal transformation carried out in the basis I. In accordance with this, the infinitesimal transformation matrix will contain the projection of the angular velocity vector ω onto the axes of the fixed basis I, i.e., in accordance with (3.49) and (3.47):

$$\Delta U = 1 + \frac{1}{2} V_I \Delta t = 1 + \frac{\Delta t}{2} \begin{vmatrix} i\Omega_3 & \Omega_2 + i\Omega_1 \\ -\Omega_2 + i\Omega_1 & -i\Omega_3 \end{vmatrix}. \qquad (3.53)$$

Proceeding from the sequence of transformations defined above, we obtain:

$$U(t+\Delta t) = U(t)\,\Delta U.$$

From this equality with consideration of (3.53) after division by Δt and taking the limit, we obtain the second form of the kinematic equation: /117

$$\dot{U} = UV_I. \qquad (3.54)$$

By multiplying this equation on the right by U+, we obtain the equation for the angular velocity matrix:

$$V_I = U^+\dot{U}. \qquad (3.55)$$

We note the formal similarity of the kinematic equation (3.54) with (3.36) and (3.12) obtained previously. We write Equation (3.54) in coordinates:

$$\begin{vmatrix} \dot{\alpha} & \dot{\beta} \\ \dot{\gamma} & \dot{\delta} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \begin{vmatrix} i\Omega_3 & \Omega_2 + i\Omega_1 \\ -\Omega_2 + i\Omega_1 & -i\Omega_3 \end{vmatrix}.$$

By carrying out the matrix multiplication and equating elements, we obtain:

$$\left.\begin{aligned} \dot{\alpha} &= \frac{i\Omega_3}{2}\alpha + \frac{i\Omega_1 - \Omega_2}{2}\beta, \quad \dot{\gamma} = \frac{i\Omega_3}{2}\gamma + \frac{i\Omega_1 - \Omega_2}{2}\delta, \\ \dot{\beta} &= -\frac{i\Omega_3}{2}\beta + \frac{\Omega_2 + i\Omega_1}{2}\alpha, \quad \dot{\delta} = -\frac{i\Omega_3}{2}\delta + \frac{\Omega_2 + i\Omega_1}{2}\gamma. \end{aligned}\right\} \qquad (3.56)$$

We note that the second pair of equations (for γ and δ) is a consequence of the first pair (the equations are complex conjugates). The Equations (3.56), exactly as for (3.52), are not yet real scalar equations, since the parameters α, β, γ and δ are complex. By separating the real and imaginary parts of (3.56) with consideration of (2.89), we obtain the equations for the quaternion (3.24).

We write out again in scalar form the equation for the components of the angular velocity in accordance with Equations (3.51) and (3.55):

$$\frac{1}{2}\begin{vmatrix} i\omega_3 & \omega_2 + i\omega_1 \\ -\omega_2 + i\omega_1 & -i\omega_3 \end{vmatrix} = \begin{vmatrix} \dot{\alpha} & \dot{\beta} \\ \dot{\gamma} & \dot{\delta} \end{vmatrix} \begin{vmatrix} \delta & -\beta \\ -\gamma & \alpha \end{vmatrix}.$$

$$\frac{1}{2}\begin{vmatrix} i\Omega_3 & \Omega_2 + i\Omega_1 \\ -\Omega_2 + i\Omega_1 & -i\Omega_3 \end{vmatrix} = \begin{vmatrix} \delta & -\beta \\ -\gamma & \alpha \end{vmatrix} \begin{vmatrix} \dot{\alpha} & \dot{\beta} \\ \dot{\gamma} & \dot{\delta} \end{vmatrix}.$$

from which we have:

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$$\left.\begin{aligned} \omega_1 &= -i(\dot{\alpha}\beta + \dot{\beta}\delta + \dot{\gamma}\alpha + \dot{\delta}\gamma), \\ \omega_2 &= \dot{\alpha}\beta + \dot{\beta}\delta - \dot{\gamma}\alpha - \dot{\delta}\gamma, \\ \omega_3 &= 2i(\dot{\gamma}\beta + \dot{\delta}\delta) = -2i(\alpha\dot{\alpha} + \beta\dot{\gamma}); \end{aligned}\right\} \qquad (3.57)$$

$$\left.\begin{aligned} \Omega_1 &= -i(\alpha\dot{\beta} + \gamma\dot{\alpha} + \beta\dot{\delta} + \delta\dot{\gamma}), \\ \Omega_2 &= \alpha\dot{\beta} + \beta\dot{\delta} - \gamma\dot{\alpha} - \delta\dot{\gamma}, \\ \Omega_3 &= 2i(\gamma\dot{\beta} + \delta\dot{\delta}) = -2i(\alpha\dot{\alpha} + \beta\dot{\gamma}). \end{aligned}\right\} \qquad (3.58)$$

One can verify by substituting (3.89) into (3.57) and (3.58) that the equalities (3.22) and (3.21) for the quaternion components are obtained. Since, in practice the real scalar kinematic equations are solved, then the equations in the Cayley-Klein parameters are identical to the equations in quaternion components, differing from them only in form. One can further show, by analogy to the kinematic equations in the direction cosines, that the forms of the kinematic equations obtained in (3.50) and (3.54) satisfy the law of addition of relative motions, with which the addition of the vectors of relative angular velocities are carried out. This proof can be carried out on the basis of the analogy of the rotation operation representations (2.80) and (1.20); all the results obtained from the quaternions being used for this.

IV. Equations for the Finite Rotation Vector. Since among the kinematic parameters there are parameters defined by the finite rotation vector (the magnitude of the rotation angle ϑ and the unit vector ζ of the finite rotation), we will derive the kinematic equations for these parameters. Since these quantities are defined by the transformation quaternion components, the kinematic equations for the quaternion are usually used for these equations. However, the equations for ϑ and ζ differ formally from the quaternion equations.

For the derivation of these equations, we use, for example, Equations (3.19), according to which:

$$\omega_E = 2\tilde{\Lambda}^{\cdot} \circ \dot{\Lambda}^{\cdot},$$

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where ω_E is the quaternion mapping the angular velocity vector onto the basis **E**, and $\Lambda^{\cdot}$ is the eigenquaternion of the transformation, i.e., the quaternion whose components are the projections of the finite rotation vector onto the basis **E**.

If the quaternion multiplication in this equation is replaced by operations in three-dimensional space, then since the vector parts of this quaternion are specified in the same basis, this equation becomes simply the vector equation. Consequently, we have, in accordance with (1.9):

$$\omega = 2(\lambda_0\dot{\lambda}_0 + \lambda_0\dot{\lambda} - \dot{\lambda}_0\lambda + \lambda\cdot\dot{\lambda} - \lambda\times\dot{\lambda}).$$

For this:

$$\lambda_0 = \cos\frac{\theta}{2},\quad \lambda = \zeta\sin\frac{\theta}{2},\quad \dot{\lambda}_0 = -\frac{\dot{\theta}}{2}\sin\frac{\theta}{2},$$

$$\dot{\lambda} = \dot{\zeta}\sin\frac{\theta}{2} + \frac{\dot{\theta}}{2}\zeta\cos\frac{\theta}{2}.$$

based on the form of the transformation (1.11), with consideration of these equalities, we obtain the equation for the angular velocity vector:

$$\omega = \dot{\theta}\zeta + \dot{\zeta}\sin\theta + \dot{\zeta}\times\zeta(1-\cos\theta). \tag{3.59}$$

The Equation (3.18) is the same operation described in another basis (I). We will leave it for the reader to show it also becomes the vector Equation (3.59).

Equation (3.59) determines the angular velocity vector through the kinematic vectors and their derivatives. To obtain the kinematic equations it is necessary to resolve (3.59) relative to the derivatives of the kinematic parameters. We take the scalar product of (3.59) with ζ. By taking into account that $\zeta\cdot\dot{\zeta}=0$, since ζ is the unit vector and $(\dot{\zeta}\times\zeta)\cdot\zeta=0$ we have:

$$\dot{\theta} = \omega\cdot\zeta. \tag{3.60}$$

To obtain the derivative of the vector ζ, we use the kinematic equation (3.11) and by method analogous to that described we convert it into the vector equation:

$$2(\dot{\lambda}_0 + \dot{\lambda}) = \lambda_0\omega - \lambda\cdot\omega + \lambda\times\omega,$$

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With consideration of the equalities expressing λ_0 and λ by θ and ζ, we obtain:

$$-\dot{\theta}\sin\frac{\theta}{2}+\dot{\theta}\zeta\cos\frac{\theta}{2}+2\dot{\zeta}\sin\frac{\theta}{2}=$$
$$=\omega\cos\frac{\theta}{2}-\omega\cdot\zeta\sin\frac{\theta}{2}+\zeta\times\omega\sin\frac{\theta}{2}.$$

By excluding the scalar part of this equality, by taking into account that

$$\omega\cos\frac{\theta}{2}-\dot{\theta}\zeta\cos\frac{\theta}{2}=\omega\cos\frac{\theta}{2}-(\omega\cdot\zeta)\zeta\cos\frac{\theta}{2}=$$
$$=\zeta\times(\omega\times\zeta)\cos\frac{\theta}{2},$$

and by dividing by $\sin\frac{\theta}{2}\neq 0$ (we will not consider $\theta=0, 2\pi$ corresponding to the identity transformation), we finally obtain:

(3.61)

One can verify that the Equation (3.12) leads to the same vector equations. The Equations (3.60) and (3.61) are the kinematic equations for the finite rotation vector.

§ 3.4. INVESTIGATION OF THE KINEMATIC EQUATIONS

The investigation carried out for the various kinematic parameters with the help of which rigid body motion can be specified have shown that there exist, generally speaking, only three different types of parameters. These are the direction cosines, the Euler angles and the quaternion components, which are the Rodriguez-Hamilton parameters. Such parameters as the finite rotation vector components and the Cayley-Klein parameters reduce to the quaternion parameters, since they define, in a different form, the same physical quantity, the equivalent Euler rotation about one axis.

The difference of the parameters is determined by the method describing the rigid body motion or, equivalently, the coordinate system coupled with it. In the case of the direction cosines, the "coordinate" position of the body coordinate system is

specified relative to the reference system; each axis of the body basis then being represented in the reference basis in a different fashion (cf. paragraph 2.1). In the case of the Euler angles, the position of the rigid body is specified with the help of three successive plane rotations such that their sum gives the equivalent position of the body basis. For the Rodriguez-Hamilton parameters, the position of the rigid body is specified by one equivalent plane rotation about some axis defined as the axis of Euler rotation. Accordingly, the kinematic equations describing the rigid body motion as a function of the magnitude of the angular velocity have, generally speaking, three different forms: the equations in the direction cosines, the Euler angles and quaternions. /121

The kinematic equations in the direction cosines are a system of nine linear scalar equations satisfying six relations, which are the orthogonality conditions; these equations do not have singular points, i.e., in principal, they are always solvable. The equations in the Euler angles (and in the Krylov angles) are a system of three independent linear equations having one singular point in which the system of parameters is degenerate. The quaternion equations are a system of four linear non-degenerate equations satisfying one coupling equation (for the quaternion norm).

Among all the kinematic parameters and, correspondingly, the methods for specifying rigid body motion, the use of quaternions is the most convenient. In contrast to the direction cosines, the orthogonal transformation operations are uniquely specified by the formal operations on quaternions (in the case of the direction cosines, as was shown in Chapter 2, there exist two different methods for specifying the motion by the orthogonal transformation matrices **A** and **B**.) In the case of quaternions, there is the minimum number of nondegenerate parameters and, correspondingly, the minimum number of relations;

this fact gives the essential advantage to these parameters when integrating numerically the kinematic equations. Finally, /122 the use of quaternions permits one to represent in one form two of the most important physical qualities characterizing rigid body motion: the characteristics of its local motion (angular velocity) and the characteristics of its position (the finite rotation vector). Both these quantities are represented by the vectors of the instantaneous Euler rotation and the equivalent Euler rotation.

The structure of quaternion algebra permits one to represent these qualities in a clear form by the vectors of the space **H**, which can always coincide with real three-dimensional space. Then the finite rotation vector and the infinitesimal rotation vector (angular velocity) have the same physical meaning. The application of another formalism leads to more complicated representations of the transformation operator and the angular velocity operator.

All this makes quaternions the most convenient kinematic parameters for investigating the general properties of rigid body motion and, in particular, for investigating the general properties of the kinematic equations. Since the kinematic equations describe the same motion, they must, of course, reduce to each other. However, this transition is not always trivial and it will be considered further. The formal representation of the kinematic equations obtained in the preceding paragraph in the general form of the multiplication of two operators (the transformation operator and the angular velocity operator) permits one to easily extend the results obtained for quaternions to the case of any kinematic parameters.

We will consider the kinematic equations for the quaternion mapping described in the form (3.11) and (3.12), where the quaternions ω_E and ω_I are specified as functions of time. In scalar

form, each of these equations is a system of four linear differential equations (3.23) and (3.24). The coefficients of these Equations are the projections ω_i of the angular velocity vector onto the body basis or the projections Ω_i onto the fixed basis, since, in the general case, the coefficients of the equations are specified functions of time.

In accordance with the general theory of linear differential equations [40], one can expect that there must exist a system of four linearly independent solutions forming the fundamental system of solutions of these equations The fundamental system can be obtained from the solutions for which, for example, the basis vectors of the four-dimensional space **H** are taken as the initial values, i.e., the vectors (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1,0), (0, 0, 0, 1). /123

With the help of the fundamental system, one can construct the general solution of the kinematic equations in the form of a linear combination of the linearly independent solutions (their number must correspond to the dimensionality of the space **H**). However, the operator representation of the kinematic Equations (3.11) and (3.12) in the form of the product of the unknown quaternion and the angular velocity quaternion permits one to find the general solution of the kinematic equations in another form. This solution is determined by the following theorem.

Theorem 3.1. Let the kinematic equation be specified in the first (3.11) or second (3.12) form for the quaternion describing the rigid body motion for a given function of time $\omega_I(t)$ or $\omega_E(t)$, and let one solution of this equation N(t) be known. Then, the quaternion:

$$\Lambda_E = G_E \circ \mathbf{N}(t) \tag{3.62}$$

is the solution of the Equation (3.11) and the quaternion:

$$\Lambda_I = \mathbf{N}(t) \circ C_I \tag{3.63}$$

is the solution of Equation (3.12) where C_E and C_I are constant quaternions. The quaternions (3.62) and (3.63) are the general solutions of the Equations (3.11) and (3.12) in the sense that any solution of these equations satisfying the required initial conditions are obtained from (3.62) and (3.63) with a corresponding selection of the constants C_I or C_E.

The proof of this theorem is obtained directly by substituting the solutions (3.62) and (3.63) into the initial Equations (3.11) and (3.12). Here, it should be taken into account that the derivatives of the quaternion products take the usual form: / 124

$$\dot{\Lambda}_E = C_E \circ \dot{\mathbf{N}}, \quad \dot{\Lambda}_I = \dot{\mathbf{N}} \circ C_I.$$

To find the solution of Equation (3.11) satisfying the initial value Λ_{E0}, it is necessary to choose the following values of the constant quaternions C_E:

$$\Lambda_{E0} = \Lambda_E(0) = C_E \circ \mathbf{N}_0,$$

from which:

$$C_E = \Lambda_{E0} \circ \tilde{\mathbf{N}}_0$$

and the solution (3.62) takes the form

$$\Lambda_E(t) = \Lambda_{E0} \circ \tilde{\mathbf{N}}_0 \circ \mathbf{N}(t). \tag{3.64}$$

By analogy, the solution of the equation (3.12) satisfying the initial condition Λ_{I0} is obtained in the form:

$$\Lambda_I(t) = \mathbf{N}(t) \circ \tilde{\mathbf{N}}_0 \circ \Lambda_{I0}. \tag{3.65}$$

One can see that $\tilde{N}_0 \circ N(t)$ is the solution of Equation (3.11) satisfying the initial condition (1,0 0,0) in precisely the same way as $N(t) \circ \tilde{N}_0$ for Equation (3.12) satisfies the same initial condition. The solutions (3.62) and (3.63) or the same in the form (3.64) and (3.65), are unique solutions satisfying the given initial conditions. This follows from the unique theorem, since $\lambda_i(t) \equiv 0$ $(i = 0, 1, 2, 3)$ are also solutions of the kinematic equations.

Let us proceed to a discussion of the theorem. Equations (3.11) and (3.12) are formally linear equations of first order, if one proceeds from the quaternion algebra (i.e., if one assumes the possibility of performing the operations of linear algebra on the set of quaternions). Thus, it is quite natural that their general solutions are written in the form (3.62) and (3.63): in the form of a particular solution multiplied by a constant (quaternion).

If one assumes the quaternions as constants, then one can see that the initial conditions (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1) will no longer be linearly independent, i.e., the solutions determined by them will also be dependent. This fact shows that the kinematic equations have one fundamental solution. /125

On the other hand, it is not difficult to show that one can arrive at the same conclusion as a result of analyzing the scalar Equations (3.23) and (3.24). One can use, for example, the representation * of the basis vectors of the hypercomplex space in the form of the following 4 x 4 matrices:

* Such a correspondence is isomorphic.

$$1=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}, \quad i_1=\begin{vmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{vmatrix},$$

$$i_2=\begin{vmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&-1&0&0\end{vmatrix}, \quad i_3=\begin{vmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{vmatrix}.$$

In this case, the particular solution considered above satisfying the initial conditions (1,0,0,0) will correspond to the fundamental system of solutions having the matrix $\mathbf{1}$ as the initial matrix.

Let us consider the physical meaning of the obtained general solution. We will always assume that both forms of the kinematic equations describe the same rigid body motion. We note first that the solution Λ_E (3.64) does not satisfy the Equation (3.12) because, by substituting the solution Λ_E into this equation, we obtain:

$$2\Lambda_{E0}\circ\tilde{N}_0\circ\dot{N}(t)=\Lambda_{E0}\circ\tilde{N}_0\circ N\circ\omega_E=\omega_I\circ\Lambda_{E0}\circ\tilde{N}_0\circ N,$$

and, since $\omega_I=N\circ\omega_E\circ\tilde{N}$, then it is evident that the obtained equality can be satisfied if one of the quaternions is zero, or if all of them are colinear, or if $\Lambda_{E0}\cdot\tilde{N}_0=1$, which is not possible in the general case. In precisely the same manner, the solution (3.65) does not satisfy, in the general case, Equation (3.11). Hence, one can conclude that each of the kinematic equations (3.11) and (3.12) has its own set of solutions $\Lambda_E(t)$ and $\Lambda_I(t)$, /126 which are determined uniquely by their own initial conditions.

As was just shown, the sets $\Lambda_E(t)$ and $\Lambda_I(t)$ do not coincide, i.e., the elements of $\Lambda_E(t)$ in the general case do not satisfy Equation (3.12), and $\Lambda_I(t)$ — Equation (3.11). It appears that the following theorem is valid.

Theorem 3.2. The sets of the solutions of (3.62) and (3.63) for the same rigid body motion intersect at one and only one point, this solution belonging to both sets is the solution N(t).

Actually, the fact that N(t) satisfies both forms of the equations was assumed from the very beginning. Conversely: each particular solution obtained from the general solution (3.62) or (3.63) is determined by the initial conditions. The solution (3.62) then satisfies Equation (3.12) only for the natural initial condition:

$$\Lambda_{E0} \circ \bar{N}_0 = 1.$$

In exactly the same way, one can verify that the solution (3.63) satisfies Equation (3.11) for:

$$\bar{N}_0 \circ \Lambda_{I0} = 1,$$

and under these conditions, the quaternion N(t) is the solution of (3.62) and (3.63).

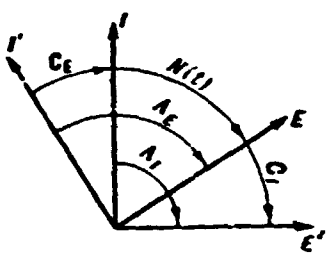

Figure 3.4.

The physical meaning of the general solutions $\Lambda_E(t)$ and $\Lambda_I(t)$ is explained in Figure 3.4, in which each basis is conditionally represented by a vector. The quaternion N(t) gives the transition from the basis I to the basis E and thereby describes the motion of the basis E relative to the basis I, which results because of the angular velocity ω. This motion can be observed from the basis I; this motion corresponds to Equation (3.12). Obviously, the motion of any other coordinate system /127 E' rotating relative to I with the same angle velocity ω will be observed from the basis I in exactly the same way, since this motion is characterized by the same projections of the vector ω onto the basis I (i.e., by the quaternion ω_I). Since the motion

of the bases **E** and **E'** occurs with the same angular velocity relative to the basis **I**, their relative position will not change and can be specified by the constant quaternion C_I. Thus, the general solution (3.63) corresponds to the description of the motion of an arbitrary basis **E** relative to the basis **I** observed from the basis **I** and specified by the function $\omega_I(t)$.

By analogy to this case, the general solution (3.62) corresponds to the motion of the basis **E** described not only relative to the reference basis **I** but relative to any other basis **I'** rotated relative to the basis **I** by a constant angle determined by the quaternion C_E. The observation of the motion then occurs from the body basis **E**; this corresponds to the Equation (3.11), since for observation from the basis **E** the motion is characterized by the projections of the angular velocity vector ω onto the basis **E**, i.e., by the quaternion ω_E.

Obviously, this motion appears the same, relative to any fixed reference basis **I'**. One should not forget that what is being considered is the motion of the bases **E** and **E'** relative to the bases **I** and **I'**, which occurs with the same specified velocity $\omega(t)$. Either of the general solutions $\Lambda_E(t)$ or $\Lambda_I(t)$ satisfied one form of the Equation (3.11) or (3.12); both forms of the equations, in accordance with theorem 3.2, are satisfied by the unique solution **N**(t).

Let us find the second form of the kinematic equation for motions described by the functions $\Lambda_I(t)$ and $\Lambda_E(t)$. We consider the solution $\Lambda_I(t)$ satisfying the Equation (3.12):

$$2\dot{\Lambda}_I = \omega_I \circ \Lambda_I.$$

Obviously, the seond form of this equation describing the motion of the basis **E'** relative to the basis **I** will be the equation containing the projection of the angular velocity vector ω

onto the basis E', i.e., the quantity $\omega_{E'}$. The mapping / 128
of the vector ω onto the basis E' is defined by the rotation operation:

$$\omega_{E'} = \tilde{\Lambda}_I \circ \omega_I \circ \Lambda_I,$$

from which we obtain the second form of the equation for the motion $\Lambda_I(t)$:

$$2\dot{\Lambda}_I = \Lambda_I \circ \omega_{E'}. \tag{3.66}$$

By analogy, we also have two forms of the equation for the quaternion $\Lambda_E(t)$.

$$2\dot{\Lambda}_E = \Lambda_E \circ \omega_E = \omega_{I'} \circ \Lambda_E, \tag{3.67}$$

where $\omega_{I'} = \Lambda_E \circ \omega_E \circ \tilde{\Lambda}_E$ is defined by the projections of the vector ω onto the basis I'. By substituting the quantities Λ_I and Λ_E from (3.63) and (3.62) for $\omega_{I'}$ and $\omega_{E'}$ in the obtained relations, we have:

$$\left.\begin{aligned} \omega_{E'} &= \tilde{C}_I \circ \tilde{N} \circ \omega_I \circ N \circ C_I = \tilde{C}_I \circ \omega_E \circ C_I, \\ \omega_{I'} &= C_E \circ N \circ \omega_E \circ \tilde{N} \circ C_E = C_E \circ \omega_I \circ C_E, \end{aligned}\right\} \tag{3.68}$$

which satisfies the known rules for projecting a fixed vector. Hence, the interpretation of theorem 3.2 follows:

If two series of time functions $\omega_I(t)$ and $\omega_E(t)$ defined by the projections of the vector $\omega(t)$ onto the bases I and E are given, then this itself uniquely determines the initial position of these two bases. On the other hand, if one form of the equations defining the projection quaternion of the angle velocity vector $\omega(t)$ and the initial position of the basis I and E are given, then this itself uniquely specifies the second form of the kinematic equation.

The relations (3.68), as well as Equations (3.66) and (3.67), serve as an example of such a relation. Hence, it also follows

that the solution $A(t)$ satisfying the two given forms of the kinematic equations (given in the sense that the actual form of the functions $\omega_i(t)$ and $\Omega_i(t)$, the projection of the angular velocity vector onto the reference and body bases are defined) is unique, since the initial conditions of this equation are uniquely defined.

One should keep in mind that one of the functions $\omega_E(t)$ and $\omega_I(t)$ can be arbitrary and uniquely determines the angular velocity vector. However, both of them related by the relation (3.13) of the orthogonal transformation, which is not arbitrary but is determined by the solution of the kinematic equation in one of the two forms.

The investigations carried out here premit one to prove another theorem related to broadening the particular solution to other cases of motion. According to theorem 3.1, one can find different solutions for the given equation, in which ω_I or ω_E are specific functions of time, from one known particular solution. However, in several cases this particular solution permits one to find a solution for other $\omega_I(t)$ or $\omega_E(t)$; these solutions are determined by the following theorem.

Theorem 3.3. Let $N(t)$ be the solution of the kinematic equation for motion with the angular velocity $\omega(t)$. In this case, for motion with the angular velocity $\omega'(t)$ obtained from $\omega(t)$ by the orthogonal transformation:

$$\omega' = S \circ \omega \circ \tilde{S}, \tag{3.69}$$

where S is a constant quaternion, there exists the particular solution obtained as:

$$N' = S \circ N \circ \tilde{S} \tag{3.70}$$

and satisfying the equation:

$$2\dot{\Lambda}' = \omega_I' \circ \Lambda' = \Lambda' \circ \omega_E'. \tag{3.71}$$

We have encountered part of the results of this theorem in the investigations of the second forms of the equations for the general solutions $\Lambda_I(t)$ and $\Lambda_E(t)$ (cf. 3.66) and (3.67); in this case, the quaternions ω_E and ω_I were present in the equations. The solutions for these angular velocity quaternions are, in turn, also solutions for the angular velocity, which has the mapping onto the basis I equaling ω_I, as well as for the other angular velocity whose mapping onto the basis E is ω_E.

We carry out the proof of theorem 3.3 for the unique solution satisfying the two given forms for the kinematic Equation (3.11) and (3.12): /130

$$2\dot{\Lambda} = \omega_I \circ \Lambda = \Lambda \circ \omega_E.$$

Obviously, if the angular velocity ω' is obtained from ω by the transformation (3.69), then the mappings onto the bases I and E link ω_I and ω_E by the same relations. Actually, if ω_I coincides with ω, then ω_I' also coincides with ω', i.e.:

$$\tilde{\omega}_I' = S \circ \omega_I \circ \tilde{S}, \quad \omega_E' = S \circ \omega_E \circ \tilde{S}.$$

By substituting (3.69) and (3.70) into Equation (3.71) we obtain:

$$2\tilde{S} \circ \dot{N} \circ S = \tilde{S} \circ \omega_I \circ N \circ S,$$

from which it is obvious that N' satisfies the first form of the equation.

We will show that N' also satisfies the second form of the equation. Obviously, it is necessary in that one has the relation:

$$\omega_E' = \tilde{\Lambda}' \circ \omega_I' \circ \Lambda',$$

which is obtained following relations:

$$\omega'_E = S \circ \omega_E \circ \tilde{S} = S \circ \tilde{N} \circ \omega_I \circ N \circ \tilde{S} =$$
$$= S \circ \tilde{N} \circ \tilde{S} \circ \omega'_I \circ S \circ N \circ \tilde{S} = \tilde{N}' \circ \omega'_I \circ N'.$$

We note that the solution $N'(t)$ satisfies the same initial condition (the relative position of the bases) as $N(t)$. The physical meaning of the theorem is rather simple: the relative angular velocity vector $\omega(t)$ can be rotated by a constant angle into a different position in the bases **I** and **E**. The corresponding solutions (3.70) then differ by a constant quaternion.

In conclusion, let us indicate another important property of the kinematic equation:

The kinematic equation for a given vector function of the angular velocity vector $\omega(t)$ can be reduced to the kinematic equation with the unit angular velocity vector: /131

$$\frac{\omega(t)}{\omega(t)}.$$

Such a possibility follows from the uniqueness of the kinematic equation; for this, it is necessary to make a substitution in Equations (3.11) and (3.12) of the independent variable:

$$d\tau = \omega(t)\,dt, \quad \omega(t) \neq 0 \tag{3.72}$$

and in accordance with this, to replace the argument according to the relation:

$$\tau = \int_0^t \omega(t)\,dt = \Phi(t), \quad t = \Phi^{-1}(\tau). \tag{3.73}$$

By substituting (3.72) and (3.73) into the kinematic Equations (3.11) and (3.12), we obtain them in the form:

$$2\frac{d\Lambda}{d\tau} = \frac{\omega_I[\Phi^{-1}(\tau)]}{\omega[\Phi^{-1}(\tau)]} \circ \Lambda = \Lambda \circ \frac{\omega_E[\Phi^{-1}(\tau)]}{\omega[\Phi^{-1}(\tau)]}. \tag{3.74}$$

Here ω_I/ω and ω_E/ω are unit vectors since the length (norm) of the vector does not change with this projection onto different bases, i.e., $|\omega_I|=\omega$ and $|\omega_E|=\omega$.

The representation of the kinematic equations in the form (3.74), where the angular velocities are unit vectors, can sometimes be convenient for their solution.

The properties of the kinematic equations considered in this paragraph are general and remain valid for any kinematic parameters. This result follows directly from the general operator form of the kinematic equations, identical to Equations (3.11) and (3.12) obtained in paragraph 3.3. Thus, all the conclusions reached can be repeated verbatim for any kinematic equation.

§ 3.5. PARTICULAR SOLUTIONS OF THE KINEMATIC EQUATIONS

I. Various methods for representing a quaternion were considered in the first chapter and it was shown, in particular, that any quaternion can be represented by an arc on the sphere [cf. (1.11)]. In view of the fact that in this representation the unit vector of the quaternion plays the role of the imaginary /132 unit, i.e., $\zeta\circ\zeta=-1$, by analogy to the well known Euler formula for complex numbers, the following representation of quaternions is possible *:

$$\Lambda=\cos\frac{\Phi}{2}+\zeta\sin\frac{\Phi}{2}=e^{\zeta\frac{\Phi}{2}}. \tag{3.75}$$

Obviously, the conjugate quaternion satisfied the relation:

$$\tilde{\Lambda}=\cos\frac{\Phi}{2}-\zeta\sin\frac{\Phi}{2}=e^{-\zeta\frac{\Phi}{2}},$$

---

* The possibility of such a representation was indicated by Hankel [12].

and the de Moivre formula considered in paragraph 1.2 will have the form, with the use of such a representation:

$$\Lambda^n = \cos n\frac{\theta}{2} + \zeta \sin n\frac{\theta}{2} = e^{\zeta n\frac{\theta}{2}}.$$

However, such a convenient representation of a quaternion in the form of an exponential function of a vector, whose length is determined by the rotation angle, has not found wide circulation, as has occurred, for example, in the case of complex numbers. The point is that the product of two quaternions cannot be represented, in the general case, in the form of an exponential function whose exponent is the sum of the exponents of the components:

$$\Lambda_1 \circ \Lambda_2 = e^{\zeta_1\frac{\theta_1}{2}} \circ e^{\zeta_2\frac{\theta_2}{2}} \neq e^{\zeta_1\frac{\theta_1}{2}+\zeta_2\frac{\theta_2}{2}}. \qquad (3.76)$$

This can be verified if only the fact that the sum of the vectors in the exponent is commutative, i.e.:

$$\zeta_1\frac{\theta_1}{2} + \zeta_2\frac{\theta_2}{2} = \zeta_2\frac{\theta_2}{2} + \zeta_1\frac{\theta_1}{2},$$

and this must reduce uniquely to the possibility of permuting the quaternion factors, which is not possible in general.

We perform the multiplication of the quaternions by representing their operations in three-dimensional space (1.9): / 133

$$\begin{aligned}\Lambda_1 \circ \Lambda_2 &= \left(\cos\frac{\theta_1}{2} + \zeta_1 \sin\frac{\theta_1}{2}\right) \circ \left(\cos\frac{\theta_2}{2} + \zeta_2 \sin\frac{\theta_2}{2}\right) = \\ &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \zeta_1 \cdot \zeta_2 \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} + \zeta_1 \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \\ &\quad + \zeta_2 \sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + \zeta_1 \times \zeta_2 \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}. \qquad (3.77)\end{aligned}$$

One can see that the operations of quaternion multiplication lead to the addition of the exponents only in two cases.

a) The vectors ζ_1 and ζ_2 are colinear. As is seen from (3.77), we have for $\zeta_1=\zeta_2=\zeta$:

$$e^{\zeta\frac{\vartheta_1}{2}}\circ e^{\zeta\frac{\vartheta_2}{2}}=e^{\zeta\frac{\vartheta_1+\vartheta_2}{2}}.$$

Geometrically, this case corresponds to the addition of the arcs of the same great circle; obviously, this addition corresponds also to the addition of colinear vectors, whose length is defined as proportional to the arc angle.

b) The angles ϑ_1 and ϑ_2 are infinitesimal quantities. As follows from (3.77), in this case the product $\sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2}\approx\frac{\vartheta_1\vartheta_2}{4}$ is a second order quantity, to which the following relation is satisfied:

$$\Lambda_1\circ\Lambda_2\approx 1+\zeta_1\frac{\vartheta_1}{2}+\zeta_2\frac{\vartheta_2}{2}=e^{\zeta_1\frac{\vartheta_1}{2}}\circ e^{\zeta_2\frac{\vartheta_2}{2}}=$$

$$=e^{\zeta_2\frac{\vartheta_2}{2}}\circ e^{\zeta_1\frac{\vartheta_1}{2}}=e^{\zeta_1\frac{\vartheta_1}{2}+\zeta_2\frac{\vartheta_2}{2}}.$$

It was indicated previously [cf. (3.8)] that for infinitesimal angles the product of quaternions gives the vector sum of infinitesimal rotation vectors. Geometrically, this means that infinitesimal arcs on the sphere add as vectors (because for small arcs the sphere differs little from the tangent plane).

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II. Let us now consider the derivative of the quaternion represented in the form (3.75). In view of the fact that the quaternion describing rigid body motion is a variable, the quantities ζ and ϑ will also be functions of time. According to the definition of the derivative, we have:

$$\dot{\Lambda}=\lim_{\Delta t\to 0}\frac{\Lambda(t+\Delta t)-\Lambda(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{e^{\frac{1}{2}\zeta(t+\Delta t)\vartheta(t+\Delta t)}-e^{\frac{1}{2}\zeta(t)\vartheta(t)}}{\Delta t}.$$

The exponent of the first exponential function can be expanded in the series:

$$\zeta(t+\Delta t)\vartheta(t+\Delta t)=\zeta(t)\vartheta(t)+\dot{\zeta}(t)\vartheta(t)\Delta t+$$

$$+\zeta(t)\dot{\vartheta}(t)\Delta t+o(\Delta t).$$

Nonetheless, as was shown above, it is impossible to represent, in the general case, an exponential function of a sum in the form of the product of exponential functions. Obviously, this is possible only in the case when the vector ζ is constant one can see that for this:

$$e^{\frac{1}{2}\zeta\Phi(t+\Delta t)}=e^{\frac{1}{2}\zeta[\Phi(t)+\dot{\Phi}\Delta t+o(\Delta t)]}=e^{\frac{1}{2}\zeta\Phi(t)}\circ e^{\frac{1}{2}\zeta[\dot{\Phi}\Delta t+o(\Delta t)]}=$$
$$=e^{\frac{1}{2}\zeta[\dot{\Phi}\Delta t+o(\Delta t)]}\circ e^{\frac{1}{2}\zeta\Phi(t)}.$$

The derivative of such a quaternion will equal:

$$\dot{\Lambda}=\lim_{\Delta t\to 0}\frac{e^{\frac{1}{2}\zeta\Phi(t)}\circ\left[e^{\frac{1}{2}\zeta[\dot{\Phi}\Delta t+o(\Delta t)]}-1\right]}{\Delta t}=$$
$$=e^{\frac{1}{2}\zeta\Phi(t)}\circ\lim_{\Delta t\to 0}\frac{e^{\frac{1}{2}\zeta[\dot{\Phi}\Delta t+o(\Delta t)]}-1}{\Delta t}=$$
$$=\lim_{\Delta t\to 0}\frac{e^{\frac{1}{2}\zeta[\dot{\Phi}\Delta t+o(\Delta t)]}-1}{\Delta t}\circ e^{\frac{1}{2}\zeta\Phi(t)}.$$

One can verify that:

$$\lim_{\Delta t\to 0}\frac{e^{\frac{1}{2}\zeta[\dot{\Phi}\Delta t+o(\Delta t)]}-1}{\Delta t}=\frac{1}{2}\zeta\dot{\Phi}$$

[for example, by representing the exponential function in the form of an expansion in sines and cosines (3.75)]. With consideration of this, we obtain finally: /135

$$\dot{\Lambda}=e^{\frac{1}{2}\zeta\Phi}\circ\frac{1}{2}\zeta\dot{\Phi}=\frac{1}{2}\zeta\dot{\Phi}\circ e^{\frac{1}{2}\zeta\Phi}. \qquad (3.78)$$

This relation indicates that in the case, when the finite rotation vector $\theta=2\zeta\operatorname{tg}\frac{\Phi}{2}$ has a constant direction, the derivative of the quaternion can be represented in the form of the product of the derivative of the exponent (3.75) and the quaternion itself:

$$\dot{\Lambda}=\frac{1}{2}\Lambda\circ\zeta\dot{\Phi}=\frac{1}{2}\zeta\dot{\Phi}\circ\Lambda.$$

Here the permutation of the factors is possible, since the quaternions ζ and Λ have colinear vector parts.

By comparing the relation (3.78) with the kinematic Equations (3.11) and (3.12), one can see that:

$$\mathbf{N}(t)=e^{\frac{1}{2}\zeta\Phi} \tag{3.79}$$

is a solution of the kinematic equation in the case when the angular velocity vector has a constant direction, then:

$$\omega(t)=\zeta\omega(t)=\zeta\dot{\Phi}(t). \tag{3.80}$$

The particular solution (3.79) must define all solutions of the kinematic equations for a given case of motion. If the angular velocity vector is specified with the help of its projections onto the basis I, then the quaternion ω_I is thereby specified:

$$\omega_I=\zeta_I\dot{\Phi},$$

since only the direction cosines of the unit vector, and not its length, are changed in the projection. In accordance with theorem 3.1, formula (3.79) determines the general solution of the Equatio (3.12) in the form:

$$\Lambda_I(t)=e^{\frac{1}{2}\zeta_I\Phi(t)}\circ C_I. \tag{3.81}$$

By analogy, if the vectors $\omega(t)$ are specified with the help of its projections in the body basis E, the general solution of the kinematic Equation (3.11) is obtained in the form: /136

$$\Lambda_E(t)=C_E\circ e^{\frac{1}{2}\zeta_E\Phi(t)}. \tag{3.82}$$

Now let the kinematic equations describe the motion of the basis E relative to the basis I occurring with the given angular velocity (3.80), the initial position of the basis E being

determined by the quaternion C_I. Since the magnitude of the rotation angle $\vartheta(t)$, according to (3.80), equals [cf. also (3.60)]:

$$\vartheta(t)=\int_0^t \omega(t')\cdot\zeta\,dt'=\int_0^t \omega(t')\,dt', \tag{3.83}$$

then at the initial moment $\vartheta(0)=0$.

Let us find the form of the solutions (3.82) and (3.81) for this case of motion; obviously, the equality:

$$\Lambda_E(t)=\Lambda_I(t).$$

must be satisfied. We have at the initial moment:

$$\Lambda_E(0)=C_E=\Lambda_I(0)=C_I,$$

from which the solution (3.82) must have the form:

$$\Lambda_E=C_I\circ e^{\frac{1}{2}\zeta_I\vartheta}.$$

Hence, it follows that Λ_I will equal Λ_E if and only if the following relation is satisfied:

$$\Lambda_I=e^{\frac{1}{2}\zeta_I\vartheta}\circ C_I=C_I\circ\left(\tilde{C}_I\circ e^{\frac{1}{2}\zeta_I\vartheta}\circ C_I\right)=C_I\circ e^{\frac{1}{2}(\tilde{C}_I\circ\zeta_I\circ C_I)\vartheta},$$

i.e.:

$$\zeta_E=\tilde{C}_I\circ\zeta_I\circ C_I,$$

exactly the same as:

$$\omega_E=\tilde{C}_I\circ\omega_I\circ C_I,$$

i.e., the form of ω_I and ω_E and of the two forms of the kinematic equation is determined only by the initial conditions.

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In the case when $C_I = 1$, the solution under consideration becomes the particular solution N(t) (3.79); if the quaternion C_I is also directed along the vector ζ, i.e., the representation:

$$C_I=e^{\frac{1}{2}\zeta_I\omega_I},$$

is possible, then we have the solution (3.81) in the form:

$$\Lambda_t = e^{\frac{1}{2}\zeta_t(\Phi+\Phi_\omega)},$$

which satisfies the initial condition of the phase shift type. The example under consideration well explains the content of the theorems of the preceding paragraph.

We indicate another possible form for representing this particular solution, when the angular velocity vector has a constant direction:

$$\mathbf{N}(t) = e^{\frac{1}{2}\int_0^t \omega(t')dt'}. \tag{3.84}$$

This form is obtained from (3.79) with consideration of (3.83). Physically, the considered solution represents the simple case of rotation in one fixed plane.

III. According to theorem 3.3, the solution for any other case of motion with an angular velocity obtained from the given constant transformation can be obtained from the known solution of the kinematic equation for the given angular velocity vector. We will try to use such an approach in the case of an arbitrary angular veloc

Obvious angular velocity vector $\boldsymbol{\omega}$ can be presented in this form:

$$\omega(t) = S(t) \circ \zeta\omega(t) \circ \tilde{S}(t) = \omega(t) S \circ \zeta \circ \tilde{S},$$

i.e., in the form of a variable unit vector whose position is determined by the quaternion S(t) and the time function $\omega(t)$. The solution for the case of the constant direction of the angular velocity vector $\zeta\omega(t)$, which is denoted by N(t) was determined above. Let us now find how the solution of the kinematic equation for the motion of $\Lambda(t)$ with the angular velocity $\omega(t)$ differs

from the quantity $\mathbf{N}'=\mathbf{S}\circ\mathbf{N}\circ\tilde{\mathbf{S}}$. We will consider the equation in projections onto the reference basis I. The equation (3.12) will be the initial equation; let us find what equation is satisfied by the quaternion N'. /138

By differentiating, we obtain:

$$\dot{\mathbf{N}}'=\dot{\mathbf{S}}\circ\mathbf{N}\circ\tilde{\mathbf{S}}+\mathbf{S}\circ\dot{\mathbf{N}}\circ\tilde{\mathbf{S}}-\mathbf{S}\circ\mathbf{N}\circ\tilde{\mathbf{S}}\circ\dot{\mathbf{S}}\circ\tilde{\mathbf{S}}.$$

Let the quaternion S satisfy the equation:

$$2\dot{\mathbf{S}}=\boldsymbol{\Omega}_I\circ\mathbf{S}.$$

By substituting the expression for the derivative in the equations N' and taking into account that the quaternion N is determined by the equation:

$$2\dot{\mathbf{N}}=\zeta_I\boldsymbol{\omega}(t)\circ\mathbf{N},$$

we obtain:

$$2\dot{\mathbf{N}}'=2(\dot{\mathbf{S}}\circ\tilde{\mathbf{S}})\circ\mathbf{S}\circ\mathbf{N}\circ\tilde{\mathbf{S}}+\mathbf{S}\circ\zeta_I\boldsymbol{\omega}(t)\circ\mathbf{N}\circ\tilde{\mathbf{S}}-$$
$$-\mathbf{S}\circ\mathbf{N}\circ\tilde{\mathbf{S}}\circ(2\dot{\mathbf{S}}\circ\mathbf{S})=(\boldsymbol{\Omega}_I\circ\mathbf{N}'-\mathbf{N}'\circ\boldsymbol{\Omega}_I)+\boldsymbol{\omega}_I(t)\circ\mathbf{N}'. \qquad (3.85)$$

Hence, it is seen that N' satisfies the same equation as $\Lambda(t)$ with an accuracy up to the term $\boldsymbol{\Omega}_I\circ\mathbf{N}'-\mathbf{N}'\circ\boldsymbol{\Omega}_I$; the solution N' can always be obtained from the known solution N by the method described. The case of $\boldsymbol{\Omega}_I\circ\mathbf{N}'=\mathbf{N}'\circ\boldsymbol{\Omega}_I$ reduces to $\boldsymbol{\Omega}_I=0$, i.e., S = const, which was also considered in theorem 3.3, or to the requirement that $\boldsymbol{\Omega}_I$ and vect N' are oarakkek,

IV. The relation (3.85) shows that in some cases of motion the solution of the kinematic equation can b·· obtained in the form of the product of quaternions, each of which satisfies the simpler kinematic equation. Let us consider this question in more detail.

Let the kinematic equation (3.11) or (3.12) be given; its solution will be found in the form of the product of two quaternions:

$$\Lambda=\Lambda_1\circ\Lambda_2,$$

which corresponds to the angular representation of the unknown transformation in the form of two successively performed transformations (for example, the transition from the basis I to to the basis I', and then to the basis E). By differentiating A we find that:

$$\dot{A}=\dot{A}_1\circ A_2+A_1\circ\dot{A}_2=\frac{1}{2}\omega_{1I}\circ A_1\circ A_2+\frac{1}{2}A_1\circ\omega_{2I'}\circ A_2=$$
$$=\frac{1}{2}(\omega_{1I}+A_1\circ\omega_{2I'}\circ\tilde{A}_1)\circ A.$$

The problem consists of selecting the resolution of the given angular velocity:

$$\omega_I(t)=\omega_{1I}+A_1\circ\omega_{2I'}\circ\tilde{A}_1=\omega_{1I}+\omega_{2I} \qquad (3.86)$$

into components ω_{1I} and $\omega_{2I}=\tilde{A}_1\circ\omega_{2I'}\circ A_1$, such that the solution is known for each of them. In particular, if ω_{1I} and $\omega_{2I'}$ have constant derivatives, then the solution $A_1(t)$ and $A_2(t)$ can be obtained in the form (3.84). We note that although ω_{1I} and $\omega_{2I'}$ will have constant directions, the resultant velocity $\omega_I(t)$ (3.86) will be a variable vector in magnitude and direction.

The method under consideration here permits one to obtain the solution of the kinematic equation of the case of finite precession. Let the angular velocity vector rotate around the circular cone about some axis, for example, i_3, such that one has the equality:

$$\omega_I=ai_1\cos bt+ai_2\sin bt+ci_3, \qquad (3.87)$$

where a, b, c are arbitrary constants.

In accordance with theorem 3.3, it is sufficient to find the solution for this specific position of the cone of precession of the vector $\omega(t)$; all of its other positions in the basis I can be obtained from the simple rotation under consideration. Let us represent this angular velocity vector in the form of the resolution (3.86), the vector $b = bi_3$ being taken as ω_{1I}.

In this case, since bi_3 determines the change of the quaternion Λ_1, we have:

$$2\dot{\Lambda}_1 = \omega_{1I} \circ \Lambda = bi_3 \circ \Lambda,$$

and Λ_1 can be represented, in accordance with (3.84), in the form:

$$\Lambda_1(t) = e^{\frac{1}{2}\int_0^t b i_3 dt'} = e^{\frac{1}{2} b i_3 t} = e^{\frac{1}{2} b t}.$$

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Further let:

$$\omega_{2r} = ai_1 + (c-b)i_3 = \omega_{I0} - b,$$

where $\omega_{I0} = \omega_I(0)$ at the initial moment of time. Then ω_{2I} is obtained from ω_{2r} by the rotation operation performed by the quaternion $\Lambda_1(t)$, i.e., we have:

$$\omega_{2I} = \Lambda_1 \circ \omega_{2r} \circ \tilde{\Lambda}_1 = e^{\frac{1}{2} b i_3 t} \circ [ai_1 + (c-b)i_3] \circ e^{-\frac{1}{2} b i_3 t} = \\ = ai_1 \cos bt + bi_2 \sin bt + (c-b)i_3.$$

Hence, it is seen that for such a selection of ω_{2r} and ω_{1I}, we actually obtain the velocity $\omega_{1I} + \omega_{2I}$ in the form (3.86) equaling ω_I (3.87).

The solution for the quaternion Λ_2 satisfying the equation:

$$2\dot{\Lambda}_2 = \omega_{2r} \circ \Lambda_2 = [ai_1 + (c-b)i_3] \circ \Lambda_2 = (\omega_{I0} - b) \circ \Lambda_2,$$

is obtained in the form (3.84), since the corresponding angular velocity vector has a fixed direction:

$$\Lambda_2 = e^{\frac{1}{2}(\omega_{I0} - b)t}.$$

Finally, we obtain the particular solution of the kinematic equation for the angular velocity vector (3.87):

$$\Lambda_I = e^{\frac{1}{2} b t} \circ e^{\frac{1}{2}(\omega_{I0} - b)t}. \qquad (3.88)$$

The solution for the other form of the kinematic equation having the angular velocity in the form (3.87) is found by analogy.

By differentiating (3.88), one can directly verify that $\Lambda_I(t)$ satisfies the kinematic Equation (3.12).

V. As was shown in paragraph 2.7, any quaternion can be expressed as a product of three quaternions having specified directions. Let us now consider the possibility of representing the solution of the kinematic equation in the form of a product of three solutions, each of which can be obtained in some fashion, for example, in the form (3.84). We consider the kinematic equation in the form (3.12) (for brevity we will omit the basis indices of the quaternion and the angular velocity).

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Let the transformation quaternion be represented in the form of the product:

$$\Lambda = \Lambda_1 \circ \Lambda_2 \circ \Lambda_3, \tag{3.89}$$

where Λ_1, Λ_2, Λ_3 specify three successive transformations, whose sum is equivalent to the given transformation. We will assume that each quaternion satisfied the equation:

$$2\dot{\Lambda}_i = \omega_i \circ \Lambda_i \qquad (i = 1, 2, 3). \tag{3.90}$$

By differentiating the relation (3.89), we obtain:

$$2\dot{\Lambda} = 2(\dot{\Lambda}_1 \circ \Lambda_2 \circ \Lambda_3 + \Lambda_1 \circ \dot{\Lambda}_2 \circ \Lambda_3 + \Lambda_1 \circ \Lambda_2 \circ \dot{\Lambda}_3) =$$
$$= (\omega_1 + \Lambda_1 \circ \omega_2 \circ \tilde{\Lambda}_1 + \Lambda_1 \circ \Lambda_2 \circ \omega_3 \circ \tilde{\Lambda}_2 \circ \tilde{\Lambda}_1) \circ \Lambda.$$

From the obtained expression, it is seen that $\Lambda(t)$ (3.89) will satisfy the initial Equation (3.12) if and only if the equality:

$$\omega(t) = \omega_1 + \Lambda_1 \circ \omega_2 \circ \tilde{\Lambda}_1 + \Lambda_1 \circ \Lambda_2 \circ \omega_3 \circ \tilde{\Lambda}_2 \circ \tilde{\Lambda}_1. \tag{3.91}$$

is satisfied. We consider the case when the components of the quaternion Λ_i are plane rotations, i.e., the vectors $\omega_i(t)$ have constant directions. As an example, we will select the following angular velocities:

$$\omega_1 = \dot{\varphi} i_3, \quad \omega_2 = \dot{\psi} i_2, \quad \omega_3 = \dot{\vartheta} i_1. \tag{3.92}$$

In this case the particular solutions of the Equations (3.90) will also be quaternions:

$$\Lambda_1 = e^{\frac{1}{2}\varphi i_3}, \quad \Lambda_2 = e^{\frac{1}{2}\psi i_2}, \quad \Lambda_3 = e^{\frac{1}{2}\vartheta i_1}. \tag{3.93}$$

Let us now ask the question: can the quantities $\varphi(t)$, $\psi(t)$ and $\vartheta(t)$ be selected such that the equality (3.91) is satisfied for any $\omega(t)$? By substituting in (3.91) the expressions for the quaternions (3.93) and the angular velocities (3.92) and by performing the quaternion multiplication operation, we obtain:

$$\omega(t) = \Omega_1 i_1 + \Omega_2 i_2 + \Omega_3 i_3 = \dot{\varphi} i_3 + \dot{\psi}(i_2 \cos\varphi - i_1 \sin\varphi) + \\ + \dot{\vartheta}(i_2 \sin\varphi \cos\psi - i_3 \sin\psi - i_1 \cos\varphi \cos\psi).$$

By equating the coefficients of the basis vectors, we find the following systems of equations for determining the angle φ, ψ and ϑ: /142

$$\Omega_1 = \dot{\vartheta} \cos\varphi \cos\psi - \dot{\psi} \sin\varphi,$$
$$\Omega_2 = \dot{\vartheta} \sin\varphi \cos\psi + \dot{\psi} \cos\varphi,$$
$$\Omega_3 = \dot{\varphi} - \dot{\vartheta} \sin\psi.$$

This system is none other than the system of kinematic equations for Krylov angles (3.42). This could have been expected since the selection of rotations (3.92) has the sequence of rotations for the Krylov angles. In exactly the same manner, another sequence (3.92) could reduce the solution of the kinematic equations to the Euler equations; the solution for the second form of the kinematic equation can also be constructed by an analogous method.

It should be noted that the representation under consideration for the sequence of plane rotations of the type (3.89) can be useful for investigating the kinematics of the motion of complex suspensions of gyroscopic systems. On the other hand, the representation of the solution of the kinematic equation in

the form (3.89) permits one to obtain a resolution of motion into a sum of more complicated known motions (3.91), and not only motions with constant directions of rotations (3.92), as was done just now.

§ 3.6. FORMAL SOLUTION OF THE KINEMATIC EQUATION

To obtain the formal solution of the kinematic equation, we will consider another possible representation of the infinitesimal transformation quaternion. In the first paragraph of this chapter, the relations (3.1) and (3.4) were presented, which relate this quaternion to the angular velocity vector. By using the representation of the quaternion (3.75), one can obtain the relation:

$$\Delta\Lambda(t)=e^{\frac{1}{2}\circ(t+\xi\Delta t)\Delta t}, \quad 0\leqslant\xi\leqslant 1. \tag{3.94}$$

In this relation it is assumed that in the time interval $[t, t+\Delta t]$ the magnitude of the angular velocity remains constant and equals its value at some average point. By comparing (3.94) /143 with the solution (3.84), one can see that the value of $\Delta\Lambda$ is obtained as the solution of the kinematic equation under the assumption that in the interval $[t, t+\Delta t]$ the magnitude of the angular velocity does not change. We have the precise value of the quaternion increment only with a transition to the limit as Δt tends to zero:

$$d\Lambda(t)=e^{\frac{1}{2}\circ(t)dt}. \tag{3.95}$$

We consider the first form of the kinematic Equation (3.11) and treat it in the form (3.9). We divide the whole time interval of the solution [0,T] into N intervals Δt_i, then:

$$\Delta t_{i+1}=t_{i+1}-t_i, \quad \max|\Delta t_i|\leqslant\frac{kT}{N} \quad (i=1, 2, \ldots, N),$$

where **N** is a finite number. In accordance with the relation (3.9), the value of the quaternion at the time t_{i+1} will be determined from its value at the time t_i with the help of the infinitesimal transformation quaternion (3.94):

$$\left.\begin{aligned}\Lambda(t_{i+1}) = \Lambda(t_i)\circ\Delta\Lambda(t_i) = \Lambda(t_i)\circ e^{\frac{1}{2}\circ\varepsilon(t_i+\xi_{i+1}\Delta t_{i+1})\Delta t_{i+1}},\\ 0\leqslant\xi\leqslant 1.\end{aligned}\right\} \qquad (3.96)$$

Here, since each quaterion is an eigenquaternion of the transformation, the angular velocity vector is a mapping of ω onto the body basis. It is seen from the relation (3.96) that if the motion begins from the quaternion Λ_0 at the time t = 0; then the value of the quaternion $\Lambda(t_{i+1})$ will be approximately represented in the form:

$$\Lambda(t_{i+1}) = \Lambda_0\circ e^{\frac{1}{2}\circ\varepsilon(\xi_1\Delta t_1)\Delta t_1}\circ e^{\frac{1}{2}\circ\varepsilon(t_1+\xi_2\Delta t_2)\Delta t_2}\circ\ldots$$
$$\ldots\circ e^{\frac{1}{2}\circ\varepsilon(t_i+\xi_{i+1}\Delta t_{i+1})\Delta t_{i+1}}. \qquad (3.97)$$ /144

The relation (3.97) is explained in Figure 3.5, on which the quaternions are represented by arcs on the sphere. We will now increase the number of intervals to infinity by decreasing the lengths of each interval. Then the solution of the equation on each interval (3.94) will tend to the exact value (3.95). One can then expect that the expression (3.97) will tend to the exact solution of the kinematic Equation (3.11).

By analogy to the Reimann integral obtained as a limit partial sums, we consider the quantities which are the limits of the products (3.97) taken in direct and reverse orders:

$$\lim_{\substack{N\to\infty\\ \Delta t_i\to 0}}\prod_{i=0}^{N} e^{\frac{1}{2}\circ\varepsilon(t_i+\xi_{i+1}\Delta t_{i+1})\Delta t_{i+1}}$$

and:

$$\lim_{\substack{N\to\infty\\ \Delta t_i\to 0}}\prod_{i=N}^{0} e^{\frac{1}{2}\circ\omega(t_i+\xi_{i+1}\Delta t_{i+1})\Delta t_{i+1}}. \qquad (3.98)$$

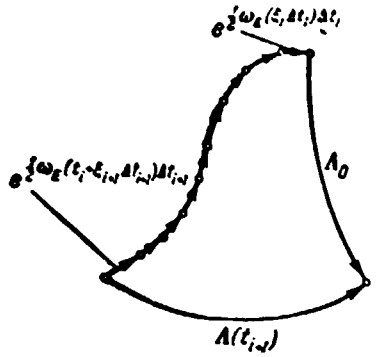

Figure 3.5.

These quantities will be denoted by the symbols:

$$\overset{T}{\underset{0}{\mathbf{S}}} e^{\frac{1}{2}\omega_E(t)\,dt} \quad \text{and} \quad \overset{0}{\underset{T}{\mathbf{S}}} e^{\frac{1}{2}\omega_I(t)\,dt}. \qquad (3.99)$$

If the limit exists, then it determines the general solution of the kinematic Equation (3.11) in the form:

$$\Lambda_E(T) = \Lambda_0 \circ \overset{T}{\underset{0}{\mathbf{S}}} e^{\frac{1}{2}\omega_E(t)\,dt}. \qquad (3.100)$$

By repeating, in an analogous manner, all the argument for the second form of the kinematic Equation (3.12), we obtain the general solution satisfying the initial condition Λ_0 in accordance with (3.99) in the form: /145

$$\Lambda_I(T) = \overset{0}{\underset{T}{\mathbf{S}}} e^{\frac{1}{2}\omega_I\,dt} \circ \Lambda_0. \qquad (3.101)$$

We will show that the functions (3.100) and (3.101) are actually solutions of the Equations (3.11) and (3.12). Let us consider the quaternion derivative as the following limit.

$$\dot{\Lambda}_E = \lim_{\Delta t \to 0} \frac{\Lambda_E(t+\Delta t) - \Lambda_E(t)}{\Delta t} =$$

$$= \Lambda_0 \circ \overset{t}{\underset{0}{\mathbf{S}}} e^{\frac{1}{2}\omega_E(t)\,dt} \lim_{\Delta t \to 0} \frac{\overset{t+\Delta t}{\underset{t}{\mathbf{S}}} e^{\frac{1}{2}\omega_E(t')\,dt'} - 1}{\Delta t}.$$

Proceeding from the definition of the quantity $\mathbf{S} e^{\frac{1}{2}\omega_E\,dt}$ as the limit of the product (3.98), one can show that:

$$\lim_{\Delta t \to 0} \frac{\overset{t+\Delta t}{\underset{t}{\mathbf{S}}} e^{\frac{1}{2}\omega_E(t')\,dt'} - 1}{\Delta t} = \frac{1}{2}\omega_E(t).$$

To find this limit, we take into account the fact that the quantity $\overset{t+\Delta t}{\underset{t}{\mathbf{S}}} e^{\frac{1}{2}\omega_E(t')\,dt'}$ can be reduced to the quantity $e^{\frac{1}{2}\int_t^{t+\Delta t}\omega_E(t')\,dt'}$, which follows from the limiting process (3.98), where for small Δt, one can replace the product of exponential functions by the

exponential function with the sum of the exponents (in accordance with the remarks of b), paragraph 3.5. The limit of the quantity:

$$\left(e^{\frac{1}{2}\int_t^{t+\Delta t}\omega_E(t')\,dt'}-1\right)\Big/\Delta t$$

is found in the usual fashion: /146

$$\lim_{\Delta t\to 0}\frac{e^{\frac{1}{2}\int_t^{t+\Delta t}\omega_E(t')\,dt'}-1}{\Delta t}=\lim_{\Delta t\to 0}\frac{e^{\frac{1}{2}\omega_E(t+\xi\Delta t)\,\Delta t}-1}{\Delta t}=\omega_E(t).$$

Hence, it follows that (3.100) is the solution of the Equation (3.11).

In finding the derivative $\dot{\Lambda}_I(t)$, one should take into account:

$$\lim_{\Delta t\to 0}\frac{\Lambda_I(t+\Delta t)-\Lambda_I(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{\overset{t}{\underset{t+\Delta t}{S}}e^{\frac{1}{2}\omega_I(t')\,dt'}-1}{\Delta t}\circ\Lambda_I(t)$$

and, by anaology to the preceding:

$$\lim_{\Delta t\to 0}\frac{\overset{t}{\underset{t+\Delta t}{S}}e^{\frac{1}{2}\omega_I(t')\,dt'}-1}{\Delta t}=\frac{1}{2}\omega_I(t),$$

i.e., (3.101) is the solution of Equation (3.12).

In conclusion, we note that in the infinite product (3.97), a as well as in the limit (3.99), the addition (i.e., integration) of the exponents of the exponential functions does not occur in general. In the particular case when the angular velocity vector has a constant direction, the summation of the exponents is possible. Obviously, one can obtain, only in this case, the expressions for the quantities (3.99):

$$\overset{t}{\underset{0}{S}}e^{\frac{1}{2}\omega_E(t')\,dt'}=e^{\frac{1}{2}\int_0^t\omega_E(t')\,dt'},\quad \overset{0}{\underset{t}{S}}e^{\frac{1}{2}\omega_I(t')\,dt'}=e^{\frac{1}{2}\int_0^t\omega_I(t')\,dt'}.$$

and then, as is not difficult to see, the solutions (3.100) and (3.101) become the solution (3.84) found previously.

CHAPTER 4

APPROXIMATE AND NUMERICAL INTEGRATION OF THE KINEMATIC EQUATIONS

The exact solution of the kinematic equations, in general, can not be expressed with functions. Thus, in practice one has to use approximation methods, similation on analog computers and numerical integration. The numerical integration of the kinematic equations in platformless control systems is of particular interest.

In this chapter we will carry out the construction and the analysis of numerical methods for integrating the kinematic equations in the body basis, i.e., Equations (3.11) Analogous construction can be carried out also for Equations (3.12).

§ 4.1. CONSTRUCTION OF THE NUMERICAL METHODS

It was shown in the preceding chapter that the general solution of Equation (3.11) can be expressed with the particular solution with unit initial conditions. This same result can be expressed in a somewhat different form. By integrating (3.11) term by term, we have

$$\Lambda(t)=\Lambda(t_0)+\frac{1}{2}\int_{t_0}^{t}\Lambda(t')\circ\omega_E(t')\,dt'. \tag{4.1}$$

We assume that:

$$\Lambda(t)=\Lambda(t_0)\circ\mathbf{N}(t). \tag{4.2}$$

By substituting the relation (4.2) into the equality (4.1), we find:

$$\mathbf{N}(t)=1+\frac{1}{2}\int\mathbf{N}(t')\circ\omega_E(t')\,dt'. \tag{4.3}$$

Thus, N(t) satisfies the kinematic equation (3.11) with the initial condition:

$$\mathbf{N}(t_0)=\mathbf{1}.$$

To construct the algorithm of the numerical integration, we assume in relation (4.2): /148

$$t_0=t_{n-1},\quad t=t_n=t_{n-1}+h,$$

where h is some interval having the sense of the increment of integration; we have:

$$\Lambda_n=\Lambda_{n-1}\circ\mathbf{N}_n. \tag{4.4}$$

In this formula we use the notation:

$$\Lambda_n=\Lambda(t_n),\quad \mathbf{N}_n=\mathbf{N}(t_n).$$

The exact solution of Equation (4.3) can be constructed by the Picard method and is the quaternion analog of the matrixant:

$$\mathbf{N}(t)=$$
$$=1+\frac{1}{2}\int_{t_0}^{t}\omega_E(t')\,dt'+\frac{1}{4}\int_{t_0}^{t}\left[\int_{t_0}^{t}\omega_E(t'')\,dt''\right]\circ\omega_E(t')\,dt'+\ldots \tag{4.5}$$

The numerical methods of various orders can be constructed on the basis of formula (4.5). It should be noted that when constructing the numerical method it is impossible, generally speaking, to take into account that the quantity $\omega_E(t)$ is easily measured at an arbitrary time. In particular, when integrating the kinematic equations in Cardanless systems, the initial information is usually treated in the form of an integral of the angular velocity over the increment of integration. Accordingly, we introduce the apparent rotation vector:

$$\boldsymbol{\theta}^{\cdot}=\int_0^t\omega_E(t')\,dt'. \tag{4.6}$$

Then, the primary information at the n-th increment is the first difference of this vector taken "backwards":

$$\nabla\boldsymbol{\theta}_n^{\cdot}=\boldsymbol{\theta}^{\cdot}(t_n)-\boldsymbol{\theta}^{\cdot}(t_{n-1})=\int_{t_{n-1}}^{t_n}\omega_E(t')\,dt'. \tag{4.7}$$

We approximate $\boldsymbol{\theta}^*(t)$ inside the increment by an interpolation polynomial of fourth order. Without reducing the generality, one can assume that at the beginning of the increment:

$$\boldsymbol{\theta}^*(t_{n-1}) = 0.$$

We have:

$$\boldsymbol{\theta}^*(\tau) \approx \left(\nabla\boldsymbol{\theta}_n^* - \frac{1}{2}\nabla^2\boldsymbol{\theta}_n^* - \frac{1}{6}\nabla^3\boldsymbol{\theta}_n^* - \frac{1}{12}\nabla^4\boldsymbol{\theta}_n^*\right)\tau + \\ + \left(\frac{1}{2}\nabla^2\boldsymbol{\theta}_n^* - \frac{1}{24}\nabla^4\boldsymbol{\theta}_n^*\right)\tau^2 + \left(\frac{1}{6}\nabla^3\boldsymbol{\theta}_n^* + \frac{1}{24}\nabla^4\boldsymbol{\theta}_n^*\right)\tau^3 + \frac{1}{24}\nabla^4\boldsymbol{\theta}_n^*\tau^4, \tag{4.8}$$

$$\frac{d\boldsymbol{\theta}^*(\tau)}{d\tau} \approx \left(\nabla\boldsymbol{\theta}_n^* - \frac{1}{2}\nabla^2\boldsymbol{\theta}_n^* - \frac{1}{6}\nabla^3\boldsymbol{\theta}_n^* - \frac{1}{12}\nabla^4\boldsymbol{\theta}_n^*\right) + \\ + \left(\nabla^2\boldsymbol{\theta}_n^* - \frac{1}{12}\nabla^4\boldsymbol{\theta}_n^*\right)\tau + \left(\frac{1}{2}\nabla^3\boldsymbol{\theta}_n^* + \frac{1}{4}\nabla^4\boldsymbol{\theta}_n^*\right)\tau^2 + \frac{1}{6}\nabla^4\boldsymbol{\theta}_n^*\tau^3, \tag{4.9}$$

where τ is the dimensional time inside the increment:

$$\tau = \frac{t - t_{n-1}}{h}.$$

By substituting the introduced magnitude of the apparent rotation vector with the help of relations (4.6), (4.8), (4.9) into formula (4.5) and keeping terms to fourth order inclusively, we obtain the following expression for the solution at the time t_n:

$$\mathbf{N}_n = 1 + \frac{1}{2}\nabla\boldsymbol{\theta}_n^* - \frac{1}{8}|\nabla\boldsymbol{\theta}_n^*|^2 + \frac{1}{24}(\nabla\boldsymbol{\theta}_n^* \times \nabla^2\boldsymbol{\theta}_n^*) - \\ - \frac{1}{48}|\nabla\boldsymbol{\theta}_n^*|^2\nabla\boldsymbol{\theta}_n^* + \frac{1}{48}(\nabla\boldsymbol{\theta}_n^* \times \nabla^3\boldsymbol{\theta}_n^*) - \\ - \frac{1}{192}|\nabla\boldsymbol{\theta}_n^*|^2\nabla^2\boldsymbol{\theta}_n^* + \frac{1}{384}|\nabla\boldsymbol{\theta}_n^*|^4 + \ldots \tag{4.10}$$

To obtain the given formula here, the quaternion multiplication operations were represented in the form of vector multiplication operations (1.9).

Leaving terms of corresponding order in the relation (4.10), we obtain the algorithms of the numerical methods: /150

$$\mathbf{N}_n^* = 1 + \tfrac{1}{2}\nabla\boldsymbol{\theta}_n^*, \tag{4.11}$$

$$\mathbf{N}_n^* = 1 + \tfrac{1}{2}\nabla\boldsymbol{\theta}_n^* - \tfrac{1}{8}|\nabla\boldsymbol{\theta}_n^*|^2, \tag{4.12}$$

$$\mathbf{N}_n^* = 1 + \tfrac{1}{2}\nabla\boldsymbol{\theta}_n^* - \tfrac{1}{8}|\nabla\boldsymbol{\theta}_n^*|^2 + \tfrac{1}{24}(\nabla\boldsymbol{\theta}_n^* \times \nabla^2\boldsymbol{\theta}_n^*) - \tfrac{1}{48}|\nabla\boldsymbol{\theta}_n^*|^2\nabla\boldsymbol{\theta}_n^*. \tag{4.13}$$

The value of the quaternion Λ calculated in the given increment is determined by the expression:

$$\Lambda_n^* = \Lambda_{n-1}^* \circ \mathbf{N}_n^*.$$

In this relation, as in (4.11) - (4.13), one asterisk denotes quantities calculated from the approximation formulae.

The methods (4.11) and (4.12) have been obtained in the literature [22, 38] and are called respectively the ordinary and modified Euler methods. The formula (4.13) determines the method of third order. The expression (4.10) itself can, of course, be considered of fourth order.

The second group of numerical methods can be constructed directly for the true rotation vector.

Let:

$$\Lambda(t) = e^{\frac{1}{2}\zeta\boldsymbol{\theta}}.$$

We introduce the true rotation vector*

$$\boldsymbol{\theta} = \zeta\boldsymbol{\theta}.$$

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* Such a vector was introduced, for example, in work [14] and is called the finite rotation vector. We will call it the true rotation vector in contrast to θ [cf. (2.52)].

By differentiating this equality and eliminating $\dot{\zeta}$ with the help of Equation (3.61), we obtain the kinematic equation for the vector $\boldsymbol{\theta}$:

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}_E \frac{\theta}{2} \operatorname{ctg} \frac{\theta}{2} + \boldsymbol{\theta} (\boldsymbol{\omega}_E \cdot \boldsymbol{\theta}) \frac{1 - \frac{\theta}{2} \operatorname{ctg} \frac{\theta}{2}}{\theta^2} + \frac{1}{2} (\boldsymbol{\theta} \times \boldsymbol{\omega}_E) \tag{4.14}$$

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The expressions $\frac{\theta}{2} \operatorname{ctg} \frac{\theta}{2}$ and $\left(1 - \frac{\theta}{2} \operatorname{ctg} \frac{\theta}{2}\right) \theta^{-2}$ are analytic functions of θ^2. Thus, to solve equation (4.14), one can use the method of small parameters [4, 34], by considering $\boldsymbol{\theta}$ and $\int \boldsymbol{\omega}_E dt$ as quantities of first order; we represent the solution in the form:

$$\boldsymbol{\theta} = \sum_{k=1}^{\infty} \boldsymbol{\theta}^{(k)}. \tag{4.15}$$

The equation for $\boldsymbol{\theta}^{(k)}$ is obtained by substituting (4.15) into the initial equation (4.14) and equating terms of the same order:

$$\dot{\boldsymbol{\theta}}^{(1)} = \boldsymbol{\omega}_E,$$
$$\dot{\boldsymbol{\theta}}^{(2)} = \frac{1}{2} \boldsymbol{\theta}^{(1)} \times \boldsymbol{\omega}_E.$$
$$\dot{\boldsymbol{\theta}}^{(3)} = -\frac{1}{12} \boldsymbol{\omega}_E |\boldsymbol{\theta}^{(1)}|^2 + \frac{1}{12} \boldsymbol{\theta}^{(1)} [\boldsymbol{\omega}_E \cdot \boldsymbol{\theta}^{(1)}] + \frac{1}{2} [\boldsymbol{\theta}^{(2)} \times \boldsymbol{\omega}_E],$$
$$\dot{\boldsymbol{\theta}}^{(4)} = -\frac{1}{8} \boldsymbol{\omega}_E [\boldsymbol{\theta}^{(1)} \cdot \boldsymbol{\theta}^{(2)}] +$$
$$+ \frac{1}{12} \boldsymbol{\theta}^{(1)} \{\boldsymbol{\theta}^{(1)} [\boldsymbol{\omega}_E \cdot \boldsymbol{\theta}^{(2)}] + \boldsymbol{\theta}^{(2)} [\boldsymbol{\omega}_E \cdot \boldsymbol{\theta}^{(1)}]\} + \frac{1}{12} [\boldsymbol{\theta}^{(3)} \times \boldsymbol{\omega}_E].$$

It will be necessary later to perform the integration of the obtained relations. By using (4.6) - (4.9), we have at time t_n under the condition that $\boldsymbol{\theta}(t_{n-1}) = 0$ to terms of fourth order inclusively:

$$\boldsymbol{\theta}_n^{(1)} = \nabla \boldsymbol{\theta}_n^*,$$
$$\boldsymbol{\theta}_n^{(2)} = \frac{1}{12} (\nabla \boldsymbol{\theta}_n^* \times \nabla^2 \boldsymbol{\theta}_n^*) + \frac{1}{24} (\nabla \boldsymbol{\theta}_n^* \times \nabla^3 \boldsymbol{\theta}_n^*),$$
$$\boldsymbol{\theta}_n^{(3)} = 0.$$

Finally, we obtained:

$$\boldsymbol{\theta}_n = \nabla \boldsymbol{\theta}_n^* + \frac{1}{12} (\nabla \boldsymbol{\theta}_n^* \times \nabla^2 \boldsymbol{\theta}_n^*) + \frac{1}{24} (\nabla \boldsymbol{\theta}_n^* \times \nabla^3 \boldsymbol{\theta}_n^*) + \ldots \tag{4.16}$$

Keeping only the first term in (4.16), we obtain the numerical method which can be called the average velocity method:

$$\mathbf{N}_n^* = e^{\frac{1}{2} \nabla \boldsymbol{\theta}_n^*}. \tag{4.17}$$

By analogy, we obtain the methods of third and fourth orders: /152

$$\mathbf{N}_n^* = e^{\frac{1}{2}\left\{\nabla\boldsymbol{\theta}_n^* + \frac{1}{12}(\nabla\boldsymbol{\theta}_n^* \times \nabla^2\boldsymbol{\theta}_n^*)\right\}}, \tag{4.18}$$

$$\mathbf{N}_n^* = e^{\frac{1}{2}\left\{\nabla\boldsymbol{\theta}_n^* + \frac{1}{12}(\nabla\boldsymbol{\theta}_n^* \times \nabla^2\boldsymbol{\theta}_n^*) + \frac{1}{24}(\nabla\boldsymbol{\theta}_n^* \times \nabla^3\boldsymbol{\theta}_n^*)\right\}}. \tag{4.19}$$

We note that the higher differences in the obtained formulae can be expressed by the first differences at preceding increments. In particular, the expressions (4.13), (4.18), (4.19) can be converted respectively to the following form:

$$\mathbf{N}_n^* = \mathbf{I} + \frac{1}{2}\nabla\boldsymbol{\theta}_n^* - \frac{1}{8}|\nabla\boldsymbol{\theta}_n^*|^2 - \frac{1}{24}(\nabla\boldsymbol{\theta}_n^* \times \nabla\boldsymbol{\theta}_{n-1}^*) - \frac{1}{48}|\nabla\boldsymbol{\theta}_n^*|^2\nabla\boldsymbol{\theta}_n^*,$$

$$\mathbf{N}_n^* = e^{\frac{1}{2}\left\{\nabla\boldsymbol{\theta}_n^* - \frac{1}{12}(\nabla\boldsymbol{\theta}_n^* \times \nabla\boldsymbol{\theta}_{n-1}^*)\right\}},$$

$$\mathbf{N}_n^* = e^{\frac{1}{2}\left\{\nabla\boldsymbol{\theta}_n^* - \frac{1}{6}(\nabla\boldsymbol{\theta}_n^* \times \nabla\boldsymbol{\theta}_{n-1}^*) + \frac{1}{24}(\nabla\boldsymbol{\theta}_n^* \times \nabla\boldsymbol{\theta}_{n-2}^*)\right\}}.$$

The third group of numerical methods is based on the use of the difference methods of the numerical Stieltjes integration [25]. With consideration of (4.6), the Equation (3.11) can be represented in the form of Shannans equation:

$$d\mathbf{\Lambda} = \frac{1}{2}\mathbf{\Lambda} \cdot d\boldsymbol{\theta}^*. \tag{4.20}$$

It can be shown that the corresponding formulae of the numerical Stieltjes integration for the Equation (4.20) has the form *:

$$\mathbf{\Lambda}_n^* = \mathbf{\Lambda}_{n-1}^* + \frac{1}{2}\mathbf{\Lambda}_{n-1}^* \circ \nabla\boldsymbol{\theta}_n^*, \tag{4.21}$$

$$\mathbf{\Lambda}_n^* = \mathbf{\Lambda}_{n-1}^* + \frac{1}{2}\mathbf{\Lambda}_{n-1}^* \circ \nabla\boldsymbol{\theta}_n^* + \frac{1}{4}\nabla\mathbf{\Lambda}_n^* \circ \nabla\boldsymbol{\theta}_n^*, \tag{4.22}$$

$$\mathbf{\Lambda}_n^* = \mathbf{\Lambda}_{n-1}^* + \frac{1}{2}\mathbf{\Lambda}_{n-1}^* \circ \nabla\boldsymbol{\theta}_n^* + \frac{1}{4}\nabla\mathbf{\Lambda}_n^* \circ \nabla\boldsymbol{\theta}_n^* + \frac{1}{24}\{\nabla\mathbf{\Lambda}_n^* \circ \nabla^2\boldsymbol{\theta}_n^* - \nabla^2\mathbf{\Lambda}_n^* \circ \nabla\boldsymbol{\theta}_n^*\} \tag{4.23}$$

---

* Here by analogy with (4.6), we use the notation:

$$\nabla\mathbf{\Lambda}_n = \mathbf{\Lambda}_n - \mathbf{\Lambda}_{n-1}.$$

and are respectively the rectangular, trapezoidal and parabolic interpolation formulae. We note that the quantities ∇A_n^* and $\nabla^2 A_n^*$ appearing in these formulae are not known at the beginning of the calculations at a given increment and they should be calculated by extrapolating the differences to the increment ahead. The extrapolation formula for (4.22) have the form: /153

$$\nabla A_n^* \approx \nabla A_{n-1}^* + \nabla^2 A_{n-1}^* + \nabla^3 A_{n-1}^*. \tag{4.24}$$

For the parabolic formula, we have:

$$\left.\begin{aligned} \nabla A_n^* &\approx \nabla A_{n-1} + \nabla^2 A_{n-1}^* + \nabla^3 A_{n-1}^* + \nabla^4 A_{n-1}^*, \\ \nabla^2 A_n^* &\approx \nabla^2 A_{n-1}^* + \nabla^3 A_{n-1}^* + \nabla^4 A_{n-1}^*. \end{aligned}\right\} \tag{4.25}$$

As in the preceding cases, the higher differences can be expressed by first differences at preceding increments. The necessity of extrapolating the the differences is a disadvantage of this group of methods.

§ 4.2. KINEMATIC EQUATIONS FOR UNNORMALIZED QUATERNIONS. CORRECTION OF THE NORM

In the numerical solutions of the kinematic equations a specific difficulty arises, which is related to the departure of the norm of the quaternion. The fact is, that in the preceding chapters it was assumed that the quaternion norm equals one, i.e.:

$$\Lambda \circ \tilde{\Lambda} = 1. \tag{4.26}$$

In particular, the operations on quaternions corresponding to orthogonal transformations were defined under this assumption. At the same time, condition (4.26) does not follow from the form of the kinematic equation. Actually, by multiplying (3.11) on the right by $\tilde{\Lambda}$, we have:

$$\dot{\Lambda} \circ \tilde{\Lambda} = \frac{1}{2} \Lambda \circ \omega_g \circ \tilde{\Lambda}. \tag{4.27}$$

Converting to the conjugate quantities, we obtain:

$$\Lambda \circ \dot{\tilde{\Lambda}} = \frac{1}{2} \Lambda \circ \tilde{\omega}_E \circ \tilde{\Lambda} = -\frac{1}{2} \Lambda \circ \omega_E \circ \tilde{\Lambda}. \tag{4.28}$$

By adding (4.27) and (4.28) term by term, we have: /154

$$\dot{\Lambda} \circ \tilde{\Lambda} + \Lambda \circ \dot{\tilde{\Lambda}} = 0,$$

i.e.:

$$\frac{d}{dt}(\Lambda \circ \tilde{\Lambda}) = 0. \tag{4.29}$$

Thus, Equation (3.11) has the first integral:

$$\Lambda \circ \tilde{\Lambda} = \text{const},$$

The actual value of the constant does not follow from the form of the kinematic equation itself, but is determined only by specifying appropriate initial conditions. In practice, because of the technique errors and rounding-off errors, the equality (4.26) stops being satisfied with the course of time, if special measures are not taken. In this connection, it is useful to develop the form of the kinematic equation for unnormalized quaternions.

Let us consider the quaternion Λ, such that:

$$\Lambda \circ \tilde{\Lambda} = \Lambda^2 > 0,$$

and the quaternion $\Lambda^{(n)}$ obtained from Λ as the result of normalization:

$$\Lambda^{(n)} = \frac{1}{\Lambda} \Lambda. \tag{4.30}$$

Let the quaternion $\Lambda^{(n)}$ describing the motion of the coordinate system satisfy the equation:

$$\dot{\Lambda}^{(n)} = \frac{1}{2} \Lambda^{(n)} \circ \omega_E. \tag{4.31}$$

By differentiating (4.30) and substituting into (4.31), we obtain the differential equation for the quaternion Λ:

$$\dot{\Lambda} = \frac{\dot{\Lambda}}{\Lambda} \Lambda + \frac{1}{2} \Lambda \circ \omega_E. \tag{4.32}$$

We denote:

$$\frac{\dot{\Lambda}}{\Lambda}=f(t). \tag{4.33}$$

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We have finally:

$$\dot{\Lambda}=f(t)\Lambda+\frac{1}{2}\Lambda\circ\omega_E. \tag{4.34}$$

Equation (4.34) is equivalent to (3.11) in the sense that the result of normalizing Equation (4.34) satisfies Equation (3.11). Thus, independently of the magnitude and the nature of the change of $\Lambda(t)>0$, the quaternions $\Lambda^{(n)}$ and Λ being solutions of these equations describe the same motion of the coordinate systems, i.e., they specify the same transformation. However, the transformation $\Lambda^{(n)}\circ(\ )\circ\tilde{\Lambda}^{(n)}$ for normalized quaternion corresponds to the transformation $\Lambda\circ(\ )\circ\Lambda^{-1}$ for the unnormalized.

We see that the addition of the term $f(t)\Lambda$ into the kinematic equation does not change the physical meaning of this equation. Equation (4.34) is, as was (3.11), the kinematic equation describing the same motion. There are no limitations on the form of the function $f(t)$. In place of the equation for the norm (4.29), we have, in view of relation (4.33), the equation:

$$\frac{d\Lambda}{dt}=f(t)\Lambda. \tag{4.35}$$

By arbitrarily choosing the form of the function $f(t)$, one can make the point $\Lambda=1$ an asymptotically stable equilibrium position of the Equation (4.35). For example, one can take:

$$f=k(1-\Lambda)$$

or:

$$f=-k\lg\Lambda,$$

but for practical reasons it is most convenient to set:

$$f=-k(\Lambda^2-1)$$

and to solve the kinematic equation in the form:

$$\dot{\Lambda}=\frac{1}{2}\Lambda\circ\omega_E-k\Lambda(\Lambda^2-1). \qquad (4.36)$$ /156

In this case, (4.35) is integrated in the final form:

$$\Lambda(t)=\left\{1-\frac{\Lambda_0^2-1}{\Lambda_0^2}e^{-2kt}\right\}^{-1/2}.$$

For $\Lambda_0>0$, $\Lambda(t)\to 1(t\to\infty)$ occurs. Thus, introducing the term $-k\Lambda(\Lambda^2-1)$ into the kinematic equation ensures an automatic correction for the norm of the solution.

§ 4.3. NUMERICAL METHODS FOR NORM CORRECTION

The results obtained in the preceding paragraph show that the departure of the norm of the solution from one does not lead to errors in constructing the coordinate system. Nonetheless, this departure is very undesirable, since it can lead to computer register overflow or, on the other hand, to its under use in the limit of a degeneracy of the solution to zero. In addition, the orthogonal transformation operations, rotation additions, etc. for unnormalized quaternions have a more complicated form in comparison with the corresponding operations for normalized quaternions. All these facts lead to the necessity of limiting the departure of the norm of the solution from one. For the numerical methods considered in paragraph 4.1, this can be achieved by periodically normalizing the solution. However, rather complicated calculation operations are then required - evolution and division.

In this paragraph a group of numerical methods based on the solution of the kinematic equation in the form (4.36) will be constructed, in which automatic correction of the norm is provided.

We set:

$$\Lambda_n^* = \Lambda_{n-1}^* \circ \{N_n^* - k(\Lambda_{n-1}^2 - 1)\}. \tag{4.37}$$

Here N_n^* is determined from one of the formulae of numerical integration of paragraph 4.1.

By taking the square of the magnitude (norm) on both sides of (4.37), we have:

$$\Lambda_n^2 = \Lambda_{n-1}^2 \{|\text{vect } N_n^*|^2 + [\text{sqal } N_n^* - k(\Lambda_{n-1}^2 - 1)]^2\}. \tag{4.38}$$ /257

When $|\text{vect } N_n^*| = \text{const}$, $\text{sqal } N_n^* = \text{const}$, the expression (4.38) determines a sequence function for Λ_n^2 with the values thusly established:

$$\left.\begin{aligned} \Lambda_\infty^2 &= 1 + \frac{1}{k}[\text{sqal } N^* \pm \sqrt{1 - |\text{vect } N^*|^2}], \\ \Lambda_\infty^2 &= 0. \end{aligned}\right\} \tag{4.39}$$

We will be interested in the value of Λ_∞^* determined by the lower sign of the first relation of (4.39), since this value is close to one or exactly equal to it. For the methods determined by the formulae (4.11) - (4.13), we have respectively:

$$\Lambda_\infty^2 \approx 1 + \frac{1}{8k}|\nabla\Theta^*|^2,$$

$$\Lambda_\infty^2 \approx 1 + \frac{1}{128k}|\nabla\Theta^*|^2,$$

$$\Lambda_\infty^2 \approx 1 - \frac{1}{192k}|\nabla\Theta^*|^2.$$

For the methods (4.17) - (4.19) using the exponential representation, we will always have:

$$\Lambda_\infty^{2*} = 1.$$

Thus, the established value of Λ_∞^2 is close to one or exactly equal to it.

To analyze the stability, let us consider the derivative $\frac{\partial \Lambda_n^2}{\partial \Lambda_{n-1}^2}$ and require that:

$$\left|\frac{\partial\Lambda_n^2}{\partial\Lambda_{n-1}^2}\right|_{\Lambda_{n-1}^2=\Lambda_\infty^2}<1. \qquad (4.40)$$

By differentiating (4.38) and substituting $\Lambda_{n-1}^2=\Lambda_\infty^2$, we have after some transformations;

$$\left.\frac{\partial\Lambda_n^2}{\partial\Lambda_{n-1}^2}\right|_{\Lambda_{n-1}^2=\Lambda_\infty^2}=1-2k\Lambda_\infty^2\sqrt{1-|\operatorname{vect}\mathbf{N}^\circ\mathbf{f}|}. \qquad (4.41)$$

The inequality (4.40) then leads to the condition:

$$0<k<\left[\Lambda_\infty^2(1-|\operatorname{vect}\mathbf{N}^\circ\mathbf{f}|)\right]^{-1}$$

or, with consideration of (4.39):

$$0<k<\frac{1}{\sqrt{1-|\operatorname{vect}\mathbf{N}^\circ\mathbf{f}|}+\sqrt{1-|\operatorname{vect}\mathbf{N}^\circ\mathbf{f}|}-\operatorname{sqal}\mathbf{N}^\circ}. \qquad (4.42)$$ /158

The inequality (4.42) will certainly be satisfied if one requires the satisfaction of the condition:

$$0<k<1.$$

It can be shown that under these conditions the other two established values of Λ_∞^2 (4.39) become unstable.

It is natural to select $k=1/2$, especially since expression (4.41) is then very close to zero, i.e., to the value for which the rate of convergence is maximum.

Let us now consider the stability of the norm correction process for an arbitrary limited $\nabla\Theta_n^*$. In this case, the sequence functions depend on n. Then, the expressions *:

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* Here and in all the succeeding formulae, it is taken that k = 1/2.

$$\Lambda_-^2 = \Lambda^2(1{,}5 - 0{,}5\Lambda^2) \tag{4.43}$$

and:

$$\Lambda_+^2 = \Lambda^2\{|\operatorname{vect} N^*|^2 + [\operatorname{sqal} N^* - 0{,}5(\Lambda^2 - 1)]^2\} \tag{4.44}$$

are the minorant and majorant sequence functions of (4.38) obtained when $|\nabla\varepsilon_i^*|$ takes a zero or its maximum value respectively. For the methods (4.17) - (4.19) the minorant and majorant have the same fixed stable point $\Lambda_\infty^2 = 1$. The sequence function itself will then obviously have the same stable fixed point.

For methods (4.11) - (4.13) the expressions (4.43) and (4.44) determine different, although close, fixed points. Thus, Λ_n^2 reduces to some limited segment containing these points. One can show that the boundaries of this segment equal:

$$\Lambda_{max}^2 = \max_{0 \le \Lambda \le \Lambda_\infty} \{\Lambda_+^2(\Lambda^2)\},$$

$$\Lambda_{min}^2 = \Lambda_-^2(\Lambda_{max}^2).$$

The limited segment for the Euler method: /159

$$1 - \frac{3}{64}|\nabla\theta^*|^4 \le \Lambda^2 \le 1 + \frac{1}{4}|\nabla\theta^*|^2.$$

of course, has the greatest length.

Now, after the stability of the correction of the norm has been proven, we convert formula (4.37) to a form convenient for evaluating the error. For this reason, we will try to eliminate Λ_{n-1}^2 by expressing it by the initial information. We set in equality (4.38):

$$\Lambda_n^2 = 1 + \varkappa_n. \tag{4.45}$$

We have:

$$\varkappa_n = (1 + \varkappa_{n-1})\left\{|\operatorname{vect} N_n^*|^2 + \left[\operatorname{sqal} N_n^* - \frac{1}{2}\varkappa_{n-1}\right]^2\right\} - 1.$$

After replacing n by n—1 and some transformations, we obtain the expression:

$$\varkappa_{n-1} = (1+\varkappa_{n-2})\left[(\mathbf{N}^{\circ}_{n-1}\circ\tilde{\mathbf{N}}^{\circ}_{n-1}) - \right.$$
$$\left. - \varkappa_{n-2}\,\mathrm{sqal}\,\mathbf{N}^{\circ}_{n-1} + \frac{1}{4}\varkappa^2_{n-2}\right] - 1,$$

from which we finally obtain:

$$\varkappa_{n-1} = [(\mathbf{N}^{\circ}_{n-1}\circ\tilde{\mathbf{N}}^{\circ}_{n-1}) - 1] + \varkappa_{n-2}[(\mathbf{N}^{\circ}_{n-1}\circ\tilde{\mathbf{N}}^{\circ}_{n-1}) - \mathrm{sqal}\,\mathbf{N}^{\circ}_{n-1}] -$$
$$- \varkappa^2_{n-2}\left[\mathrm{sqal}\,\mathbf{N}^{\circ}_{n-1} - \frac{1}{4}\right] + \frac{1}{4}\varkappa^3_{n-2}. \quad (4.46) \qquad (4.46)$$

In order to contract the recursion relation (4.46), we replace $\varkappa_{n-2}$ calculated by the same relation. Then, by analogy, we exclude $\varkappa_{n-3}$, etc.

To terms of fourth order inclusively, we have respectively for the methods (4.11) - (4.13):

$$\varkappa_{n-1} \approx \frac{1}{4}|\nabla\boldsymbol{\theta}^{\circ}_{n-1}|^2 + \frac{1}{16}|\nabla\boldsymbol{\theta}^{\circ}_{n-1}|^2|\nabla\boldsymbol{\theta}^{\circ}_{n-2}|^2 - \frac{3}{64}|\nabla\boldsymbol{\theta}^{\circ}_{n-2}|^4, \qquad (4.47)$$

$$\varkappa_{n-1} \approx \frac{1}{64}|\nabla\boldsymbol{\theta}^{\circ}_{n-1}|^4, \qquad (4.48)$$

$$\varkappa_{n-1} \approx \frac{1}{192}|\nabla\boldsymbol{\theta}^{\circ}_{n-1}|^4. \qquad (4.49)$$

By using the equalities (4.47) - (4.49) with consideration of (4.45), one can exclude Λ^2_{n-1} from the formula (4.37).

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Thus, for the cited methods, one can replace (4.37) by the expression:

$$\mathbf{\Lambda}^{\circ}_{n} = \mathbf{\Lambda}^{\circ}_{n-1}\circ\mathbf{N}^{\circ\circ}_{n}, \qquad (4.50)$$

where respectively:

$$\mathbf{N}^{\circ\circ}_{n} = 1 + \frac{1}{2}\nabla\boldsymbol{\theta}^{\circ}_{n} - \frac{1}{8}|\nabla\boldsymbol{\theta}^{\circ}_{n-1}|^2 -$$
$$- \frac{1}{32}|\nabla\boldsymbol{\theta}^{\circ}_{n-1}|^2|\nabla\boldsymbol{\theta}^{\circ}_{n-2}|^2 + \frac{3}{64}|\nabla\boldsymbol{\theta}^{\circ}_{n-1}|^4,$$

$$\mathbf{N}^{\circ\circ}_{n} = 1 + \frac{1}{2}\nabla\boldsymbol{\theta}^{\circ}_{n} - \frac{1}{8}|\nabla\boldsymbol{\theta}^{\circ}_{n}|^2 - \frac{1}{128}|\nabla\boldsymbol{\theta}^{\circ}_{n-1}|^4,$$

$$\mathbf{N}^{\circ\circ}_{n} = 1 + \frac{1}{2}\nabla\boldsymbol{\theta}^{\circ}_{n} - \frac{1}{8}|\nabla\boldsymbol{\theta}^{\circ}_{n}|^2 + \frac{1}{24}(\nabla\boldsymbol{\theta}^{\circ}_{n}\times\nabla^2\boldsymbol{\theta}^{\circ}_{n}) -$$
$$- \frac{1}{48}|\nabla\boldsymbol{\theta}^{\circ}_{n}|^2\nabla\boldsymbol{\theta}^{\circ}_{n} + \frac{1}{384}|\nabla\boldsymbol{\theta}^{\circ}_{n-1}|^4.$$

For the methods (4.17) - (4.19) to terms of fourth order, we obtain:

$$\varkappa_{n-1} \approx 0.$$

Thus, for the method (4.17) - (4.19), the expressions for N^{**}_n and N^{*}_n coincide.

§ 4.4. ERRORS IN THE NUMERICAL METHODS WITH CONSTANT INCREMENTS

We denote by $\Delta\Lambda_n$ the difference between the values of Λ_n calculated from the approximation and the exact formulae:

$$\Delta\Lambda_n = \Lambda^*_n - \Lambda_n.$$

By comparing (4.4) with (4.50) and assuming that the preceding value Λ^*_{n-1} was exact, we have:

$$\Delta\Lambda_n = \Lambda_{n-1} \circ \delta N_n.$$

Here:

$$\delta N_n = N^*_n - N_n \tag{4.51}$$

is the error per increment of integration for the methods without correction of the norm. Accordingly, for the methods with correction of the norm:

$$\delta N_n = N^{**}_n - N_n. \tag{4.52}$$

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The explicit form of δN_n for the various numerical methods is presented in Table 4.1.

For evaluating the error accumulated over the interval of integration, we represent Λ_n in the form:

$$\Lambda_n = \Lambda_0 \circ N_1 \circ \ldots \circ N_{n-1} \circ N_n; \tag{4.53}$$

by analogy, we represent Λ^*_n with consideration of (4.51) and (4.52):

$$\Lambda^*_n = \Lambda_0 \circ (N_1 + \delta N_1) \circ \ldots \circ (N_{n-1} + \delta N_{n-1}) \circ (N_n + \delta N_n). \tag{4.54}$$

For the quaternion with the accumulated error Λ^*_n relative to Λ_n, we have:

$$\delta\Lambda_n = \Lambda^*_n \circ \tilde{\Lambda}_n. \tag{4.55}$$

Since Λ_n is a normalized quaternion, then:

$$|\delta\Lambda_n| = |\Lambda_n^*| = \prod_{k=1}^{n} N_k^*, \tag{4.56}$$

where:

$$N_k^* = |N_k^*|.$$

For the accumulated error of the magnitude, we obtain the expression:

$$\delta\Lambda_n^* = \prod_{k=1}^{n} N_k^* - 1 = \prod_{k=1}^{n} (1 + \delta N_k) - 1 = \sum_{k=1}^{n} \delta N_k + O(\delta N_k^2). \tag{4.57}$$

Here we denote by δN_n the error of the magnitude of N_n^* per given increment. The explicit form of δN_n for the different numerical methods is presented in Table 4.2.

To terms of higher order, one can replace the sum in the expression (4.57) by the integral:

$$\delta\Lambda^* \approx \frac{1}{h} \int_0^t \delta N(t')\, dt'. \tag{4.58}$$

Explicit expressions for the accumulated error of the magnitude of $\delta\Lambda^*$ and the rate of its departure are presented in Table 4.3. The fact that: /168

$$\nabla^k \Theta_n^* \approx h^k \frac{d^{k-1}}{dt^{k-1}} (\omega_E).$$

has been taken into account. We note that the component of the accumulated error determined by formula (4.58) does not lead to errors in determining the coordinate system (cf. paragraph 4.2) and can be eliminated by periodic normalization of Λ^*.

Let us now consider the component of the accumulated error which cannot be eliminated by normalization. By normalizing (4.55), we have:

$$\delta\Lambda^{*)} = \frac{1}{\Lambda_n^*} \delta\Lambda_n = \frac{1}{\Lambda_n^*} \Lambda_n^* \circ \tilde{\Lambda}_n. \tag{4.59}$$

TABLE 4.1. ERROR OF THE NUMERICAL METHODS PER INTEGRATION INCREMENT

/162-3

Numerical method	δN_n
Euler method	
$\Lambda_n = \Lambda_{n-1} \circ \left(1 + \frac{1}{2} \nabla\theta_n^*\right)$	$\frac{1}{8} \lvert \nabla\theta_n^* \rvert^2$
Modified Euler method	
$\Lambda_n = \Lambda_{n-1} \circ \left(1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8} \lvert \nabla\theta_n^* \rvert^2\right)$	$-\frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{48} \lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*$
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8} \lvert \nabla\theta_n^* \rvert^2 + \frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*) - \frac{1}{48} \lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*\right]$	$-\frac{1}{48} (\nabla\theta_n^* \times \nabla^3\theta_n^*) + \frac{1}{192} \lvert \nabla\theta_n^* \rvert^2 \nabla^2\theta_n^* - \frac{1}{384} \lvert \nabla\theta_n^* \rvert^4$
Average velocity method	
$\Lambda_n = \Lambda_{n-1} \circ e^{\frac{1}{2} \nabla\theta_n^*}$	$-\frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*)$
$\Lambda_n = \Lambda_{n-1} \circ e^{\frac{1}{2} \left[\nabla\theta_n^* + \frac{1}{12} (\nabla\theta_n^* \times \nabla^2\theta_n^*)\right]}$	$-\frac{1}{48} (\nabla\theta_n^* \times \nabla^3\theta_n^*)$
Euler method with norm correction	
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* + \frac{1}{2} (1 - \Lambda_{n-1}^2)\right]$	$-\frac{1}{4} \nabla\theta_n^* \cdot \nabla^2\theta_n^* - \frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{48} \lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*$
Modified Euler method with norm correction	
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8} \lvert \nabla\theta_n^* \rvert^2 + \frac{1}{2} (1 - \Lambda_{n-1}^2)\right]$	$-\frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{48} \lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*$
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8} \lvert \nabla\theta_n^* \rvert^2 + \frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*) - \frac{1}{48} \lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^* + \frac{1}{2} (1 - \Lambda_{n-1}^2)\right]$	$-\frac{1}{48} (\nabla\theta_n^* \times \nabla^3\theta_n^*) + \frac{1}{192} \lvert \nabla\theta_n^* \rvert^2 \nabla^2\theta_n^*$
Average velocity method with norm correction	
$\Lambda_n = \Lambda_{n-1} \circ \left[e^{\frac{1}{2} \nabla\theta_n^*} + \frac{1}{2} (1 - \Lambda_{n-1}^2)\right]$	$-\frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*)$
$\Lambda_n = \Lambda_{n-1} \circ \left[e^{\frac{1}{2} \nabla\theta_n^* + \frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*)} + \frac{1}{2} (1 - \Lambda_{n-1}^2)\right]$	$-\frac{1}{48} (\nabla\theta_n^* \times \nabla^3\theta_n^*)$
$\Lambda_n = \Lambda_{n-1} + \frac{1}{2} \Lambda_{n-1} \circ \nabla\theta_n^*$	$\frac{1}{8} \lvert \nabla\theta_n^* \rvert^2$
$\Lambda_n = \Lambda_{n-1} + \frac{1}{2} \Lambda_{n-1} \circ \nabla\theta_n^* + \frac{1}{4} \nabla\Lambda_{n-1} \circ \nabla\theta_n^*$	$-\frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{96} \lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*$
$\nabla\Lambda_n = \nabla\Lambda_{n-1} + \nabla^2\Lambda_{n-1} + \nabla^3\Lambda_{n-1}$	
$\Lambda_n = \Lambda_{n-1} + \frac{1}{2} \Lambda_{n-1} \circ \nabla\theta_n^* + \frac{1}{4} \nabla\Lambda_n \circ \nabla\theta_n^* + \frac{1}{24} (\nabla\Lambda_n \circ \nabla^2\theta_n^* - \nabla^2\Lambda_n \circ \nabla\theta_n^*)$	$-\frac{1}{48} (\nabla\theta_n^* \times \nabla^3\theta_n^*) + \frac{1}{96} \lvert \nabla\theta_n^* \rvert^2 \nabla^2\theta_n^* - \frac{1}{128} \lvert \nabla\theta_n^* \rvert^4$
$\nabla\Lambda_n = \nabla\Lambda_{n-1} + \nabla^2\Lambda_{n-1} + \nabla^3\Lambda_{n-1} + \nabla^4\Lambda_{n-1}$	
$\nabla^2\Lambda_n = \nabla^2\Lambda_{n-1} + \nabla^3\Lambda_{n-1} + \nabla^4\Lambda_{n-1}$	

TABLE 4.2. MAGNITUDE ERROR PER INTEGRATION INCREMENT /164-5

Numerical method	Magnitude error δN_n
$\Lambda_n = \Lambda_{n-1} \circ \left(1 + \frac{1}{2} \nabla\theta_n^*\right)$	$\frac{1}{8} \lvert \nabla\theta_n^* \rvert^2$
$\Lambda_n = \Lambda_{n-1} \circ \left(1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8} \lvert \nabla\theta_n^* \rvert^2\right)$	$\frac{1}{128} \lvert \nabla\theta_n^* \rvert^4$
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8} \lvert \nabla\theta_n^* \rvert^2 + \frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*) - \frac{1}{48} \lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*\right]$	$-\frac{1}{384} \lvert \nabla\theta_n^* \rvert^4$
$\Lambda_n = \Lambda_{n-1} \circ e^{\frac{1}{2} \nabla\theta_n^*}$	–
$\Lambda_n = \Lambda_{n-1} \circ e^{\frac{1}{2} \left[\nabla\theta_n^* + \frac{1}{12} (\nabla\theta_n^* \times \nabla^2\theta_n^*)\right]}$	–
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* + \frac{1}{2} (1 - \Lambda_{n-1}^2)\right]$	$\frac{1}{8} (\lvert \nabla\theta_n^* \rvert^2 - \lvert \nabla\theta_{n-1}^* \rvert^2)$
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8} \lvert \nabla\theta_n^* \rvert^2 + \frac{1}{2} (1 - \Lambda_{n-1}^2)\right]$	$\frac{1}{128} (\lvert \nabla\theta_n^* \rvert^4 - \lvert \nabla\theta_{n-1}^* \rvert^4)$
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8} \lvert \nabla\theta_n^* \rvert^2 + \frac{1}{24} (\nabla\theta_n^* \times \nabla^2\theta_n^*) - \frac{1}{48} \lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^* + \frac{1}{2} (1 - \Lambda_{n-1}^2)\right]$	$-\frac{1}{384} (\lvert \nabla\theta_n^* \rvert^4 - \lvert \nabla\theta_{n-1}^* \rvert^4)$
$\Lambda_n = \Lambda_{n-1} \circ \left[e^{\frac{1}{2} \nabla\theta_n^*} + \frac{1}{2} (1 - \Lambda_{n-1}^2)\right]$	–
$\Lambda_n = \Lambda_{n-1} \circ \left\{e^{\frac{1}{2} \left[\nabla\theta_n^* + \frac{1}{12} (\nabla\theta_n^* \times \nabla^2\theta_n^*)\right]} + \frac{1}{2} (1 - \Lambda_{n-1}^2)\right\}$	–
$\Lambda_n = \Lambda_{n-1} + \frac{1}{2} \Lambda_{n-1} \circ \nabla\theta_n^*$	$\frac{1}{8} \lvert \nabla\theta_n^* \rvert^2$
$\Lambda_n = \Lambda_{n-1} + \frac{1}{2} \Lambda_{n-1} \circ \nabla\theta_n^* + \frac{1}{4} \nabla\Lambda_n \circ \nabla\theta_n^*$ $\nabla\Lambda_n = \nabla\Lambda_{n-1} + \nabla^2\Lambda_{n-1} + \nabla^3\Lambda_{n-1}$	$\frac{5}{384} \lvert \nabla\theta_n^* \rvert^4$
$\Lambda_n = \Lambda_{n-1} + \frac{1}{2} \Lambda_{n-1} \circ \nabla\theta_n^* + \frac{1}{4} \nabla\Lambda_n \circ \nabla\theta_n^* + \frac{1}{24} (\nabla\Lambda_n \circ \nabla^2\theta_n^* - \nabla^2\Lambda_n \circ \nabla\theta_n^*)$ $\nabla\Lambda_n = \nabla\Lambda_{n-1} + \nabla^2\Lambda_{n-1} + \nabla^3\Lambda_{n-1} + \nabla^4\Lambda_{n-1}$ $\nabla^2\Lambda_n = \nabla^2\Lambda_{n-1} + \nabla^3\Lambda_{n-1} + \nabla^4\Lambda_{n-1}$	$\frac{5}{192} \lvert \nabla\theta_n^* \rvert^4$

TABLE 4.3. ACCUMULATED MAGNITUDE ERROR AND RATE OF DEPARTURE

/166-7

Numerical method	Magnitude error δA^*	Rate of departure $\frac{d\delta A^*}{dt}$
$A_n = A_{n-1}\cdot\left(1+\frac{1}{2}\nabla\theta_n^*\right)$	$\frac{h}{8}\int_0^t \omega^2\,dt'$	$\frac{h}{8}\omega^2$
$A_n = A_{n-1}\cdot\left(1+\frac{1}{2}\nabla\theta_n^* - \frac{1}{8}\lvert\nabla\theta_n^*\rvert^2\right)$	$\frac{h^3}{128}\int_0^t \omega^4\,dt'$	$\frac{h^3}{128}\omega^4$
$A_n = A_{n-1}\cdot\left[1+\frac{1}{2}\nabla\theta_n^* - \frac{1}{8}\lvert\nabla\theta_n^*\rvert^2 + \frac{1}{24}(\nabla\theta_n^*\times\nabla^2\theta_n^*) - \frac{1}{48}\lvert\nabla\theta_n^*\rvert^2\nabla\theta_n^*\right]$	$-\frac{h^5}{384}\int_0^t \omega^6\,dt'$	$-\frac{h^5}{384}\omega^6$
$A_n = A_{n-1}\cdot e^{\cdots}$	—	—
$A_n = A_{n-1}\cdot e^{\frac{1}{2}\lvert\nabla\theta_n^* + \frac{1}{12}(\nabla\theta_n^*\times\nabla^2\theta_n^*)\rvert}$	—	—
$A_n = A_{n-1}\cdot\left[1+\frac{1}{2}\nabla\theta_n^* + \frac{1}{2}\left(1-A_{n-1}^2\right)\right]$	$\frac{1}{8}h^2\omega^2$	—
$A_n = A_{n-1}\cdot\left[1+\frac{1}{2}\nabla\theta_n^* - \frac{1}{8}\lvert\nabla\theta_n^*\rvert^2 + \frac{1}{2}\left(1-A_{n-1}^2\right)\right]$	$\frac{1}{128}h^4\omega^4$	—
$A_n = A_{n-1}\cdot\left[1+\frac{1}{2}\nabla\theta_n^* - \frac{1}{8}\lvert\nabla\theta_n^*\rvert^2 + \frac{1}{24}(\nabla\theta_n^*\times\nabla^2\theta_n^*) - \frac{1}{48}\lvert\nabla\theta_n^*\rvert^2\nabla\theta_n^* + \frac{1}{2}\left(1-A_{n-1}^2\right)\right]$	$-\frac{1}{384}h^4\omega^4$	—
$A_n = A_{n-1}\cdot\left[e^{\frac{1}{2}\nabla\theta_n^*} + \frac{1}{2}\left(1-A_{n-1}^2\right)\right]$	—	—
$A_n = A_{n-1}\cdot\left\{e^{\frac{1}{2}\left\lvert\nabla\theta_n^* + \frac{1}{12}(\nabla\theta_n^*\times\nabla^2\theta_n^*)\right\rvert} + \frac{1}{2}\left(1-A_{n-1}^2\right)\right\}$	—	—
$A_n = A_{n-1} + \frac{1}{2}A_{n-1}\cdot\nabla\theta_n^*$	$\frac{h}{8}\int_0^t \omega^2\,dt'$	$\frac{h}{8}\omega^2$
$A_n = A_{n-1} + \frac{1}{2}A_{n-1}\cdot\nabla\theta_n^* + \frac{1}{4}\nabla A_n\cdot\nabla\theta_n^*$ $\nabla A_n = \nabla A_{n-1} + \nabla^2 A_{n-1} + \nabla^3 A_{n-1}$	$\frac{5}{384}h^3\int_0^t \omega^4\,dt'$	$\frac{5}{384}h^3\omega^4$
$A_n = A_{n-1} + \frac{1}{2}A_{n-1}\cdot\nabla\theta_n^* + \frac{1}{4}\nabla A_n\cdot\nabla\theta_n^* + \frac{1}{24}(\nabla A_n\cdot\nabla^2\theta_n^* - \nabla^2 A_n\cdot\nabla\theta_n^*)$ $\nabla A_n = \nabla A_{n-1} + \nabla^2 A_{n-1} + \nabla^3 A_{n-1} + \nabla^4 A_{n-1}$ $\nabla^2 A_n = \nabla^2 A_{n-1} + \nabla^3 A_{n-1} + \nabla^4 A_{n-1}$	$\frac{5}{192}h^5\int_0^t \omega^6\,dt'$	$\frac{5}{192}h^5\omega^6$

TABLE 4.4. ERROR PER INCREMENT NOT ELIMINATED BY NORMALIZATION

/170-1

Numerical method	$\delta N_n^{(n)}$
$\Lambda_n = \Lambda_{n-1} \circ \left(1 + \frac{1}{2} \nabla\theta_n^*\right)$	$-\frac{1}{24}(\nabla\theta_n^* \times \nabla^2\theta_n^*) - \frac{1}{24}\lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*$
$\Lambda_n = \Lambda_{n-1} \circ \left(1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8}\lvert \nabla\theta_n^* \rvert^2\right)$	$-\frac{1}{24}(\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{48}\lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*$
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8}\lvert \nabla\theta_n^* \rvert^2 + \frac{1}{24}(\nabla\theta_n^* \times \nabla^2\theta_n^*) - \frac{1}{48}\lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*\right]$	$-\frac{1}{48}(\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{192}\lvert \nabla\theta_n^* \rvert^2 \nabla^2\theta_n^*$
$\Lambda_n = \Lambda_{n-1} \circ e^{\frac{1}{2}\nabla\theta_n^*}$	$-\frac{1}{24}(\nabla\theta_n^* \times \nabla^2\theta_n^*)$
$\Lambda_n = \Lambda_{n-1} \circ e^{\frac{1}{2}\left[\nabla\theta_n^* + \frac{1}{12}(\nabla\theta_n^* \times \nabla^2\theta_n^*)\right]}$	$-\frac{1}{48}(\nabla\theta_n^* \times \nabla^2\theta_n^*)$
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* + \frac{1}{2}(1 - \Lambda_{n-1}^2)\right]$	$-\frac{1}{24}(\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{48}\lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*$
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8}\lvert \nabla\theta_n^* \rvert^2 + \frac{1}{2}(1 - \Lambda_{n-1}^2)\right]$	$-\frac{1}{24}(\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{48}\lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*$
$\Lambda_n = \Lambda_{n-1} \circ \left[1 + \frac{1}{2} \nabla\theta_n^* - \frac{1}{8}\lvert \nabla\theta_n^* \rvert^2 + \frac{1}{24}(\nabla\theta_n^* \times \nabla^2\theta_n^*) - \frac{1}{48}\lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^* + \frac{1}{2}(1 - \Lambda_{n-1}^2)\right]$	$-\frac{1}{48}(\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{192}\lvert \nabla\theta_n^* \rvert^2 \nabla^2\theta_n^*$
$\Lambda_n = \Lambda_{n-1} \circ \left[e^{\frac{1}{2}\nabla\theta_n^*} + \frac{1}{2}(1 - \Lambda_{n-1}^2)\right]$	$-\frac{1}{24}(\nabla\theta_n^* \times \nabla^2\theta_n^*)$
$\Lambda_n = \Lambda_{n-1} \circ \left\{e^{\frac{1}{2}\left[\nabla\theta_n^* + \frac{1}{12}(\nabla\theta_n^* \times \nabla^2\theta_n^*)\right]} + \frac{1}{2}(1 - \Lambda_{n-1}^2)\right\}$	$-\frac{1}{48}(\nabla\theta_n^* \times \nabla^2\theta_n^*)$
$\Lambda_n = \Lambda_{n-1} + \frac{1}{2}\Lambda_{n-1} \circ \nabla\theta_n^*$	$-\frac{1}{24}(\nabla\theta_n^* \times \nabla^2\theta_n^*) - \frac{1}{24}\lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*$
$\Lambda_n = \Lambda_{n-1} + \frac{1}{2}\Lambda_{n-1} \circ \nabla\theta_n^* + \frac{1}{4}\nabla\Lambda_n \circ \nabla\theta_n^*$ $\nabla\Lambda_n = \nabla\Lambda_{n-1} + \nabla^2\Lambda_{n-1} + \nabla^3\Lambda_{n-1}$	$-\frac{1}{24}(\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{96}\lvert \nabla\theta_n^* \rvert^2 \nabla\theta_n^*$
$\Lambda_n = \Lambda_{n-1} + \frac{1}{2}\Lambda_{n-1} \circ \nabla\theta_n^* + \frac{1}{4}\nabla\Lambda_n \circ \nabla\theta_n^* + \frac{1}{24}(\nabla\Lambda_n \circ \nabla^2\theta_n^* - \nabla^2\Lambda_n \circ \nabla\theta_n^*)$ $\nabla\Lambda_n = \nabla\Lambda_{n-1} + \nabla^2\Lambda_{n-1} + \nabla^3\Lambda_{n-1} + \nabla^4\Lambda_{n-1}$ $\nabla^2\Lambda_n = \nabla^2\Lambda_{n-1} + \nabla^3\Lambda_{n-1} + \nabla^4\Lambda_{n-1}$	$-\frac{1}{48}(\nabla\theta_n^* \times \nabla^2\theta_n^*) + \frac{1}{96}\lvert \nabla\theta_n^* \rvert^2 \nabla^2\theta_n^* - \frac{3}{128}\lvert \nabla\theta_n^* \rvert^4$

TABLE 4.5. ACCUMULATED POSITION ERROR

AND RATE OF DEPARTURE

/172-3

Numerical method	Accumulated position error $\delta\theta$	Rate of departure $\left\lvert\frac{d\,\delta\theta}{dt}\right\rvert$
$A_n = A_{n-1}\cdot\left(1+\frac{1}{2}\nabla\theta_n^*\right)$	$-\frac{h^2}{12}\int_0^t A\cdot(\omega\times\dot{\omega})\cdot\tilde{A}\,dt' -$ $-\frac{h^2}{12}\int_0^t A\cdot(\omega^2\omega)\cdot\tilde{A}\,dt'$	$\frac{h^2}{12}\sqrt{\lvert\omega\times\dot{\omega}\rvert^2+\omega^6}$
$A_n = A_{n-1}\cdot\left(1+\frac{1}{2}\nabla\theta_n^*-\frac{1}{8}\lvert\nabla\theta_n^*\rvert^2\right)$	$-\frac{h^2}{12}\int_0^t A\cdot(\omega\times\dot{\omega})\cdot\tilde{A}\,dt' +$ $+\frac{h^2}{24}\int_0^t A\cdot(\omega^2\omega)\cdot\tilde{A}\,dt'$	$\frac{h^2}{12}\sqrt{\lvert\omega\times\dot{\omega}\rvert^2+\frac{1}{4}\omega^6}$
$A_n = A_{n-1}\cdot\left[1+\frac{1}{2}\nabla\theta_n^*-\frac{1}{8}\lvert\nabla\theta_n^*\rvert^2+\right.$ $\left.+\frac{1}{24}\left(\nabla\theta_n^*\times\nabla^2\theta_n^*\right)-\frac{1}{48}\lvert\nabla\theta_n^*\rvert^2\nabla\theta_n^*\right]$	$\frac{h^3}{24}\int_0^t A\cdot(\omega\times\ddot{\omega})\cdot\tilde{A}\,dt +$ $+\frac{h^3}{96}\int_0^t A\cdot(\omega^2\dot{\omega})\cdot\tilde{A}\,dt'$	$\frac{h^3}{24}\left\lvert(\omega\times\ddot{\omega})-\frac{1}{4}\omega^2\dot{\omega}\right\rvert$
$A_n = A_{n-1}\cdot e^{\frac{1}{2}\nabla\theta_n^*}$	$-\frac{h^2}{12}\int_0^t A\cdot(\omega\times\dot{\omega})\cdot\tilde{A}\,dt'$	$\frac{h^2}{12}\lvert\omega\times\dot{\omega}\rvert$
$A_n = A_{n-1}\cdot e^{\frac{1}{2}\left[\nabla\theta_n^*+\frac{1}{12}\left(\nabla\theta_n^*\times\nabla^2\theta_n^*\right)\right]}$	$-\frac{h^3}{24}\int_0^t A\cdot(\omega\times\ddot{\omega})\cdot\tilde{A}\,dt'$	$\frac{h^3}{24}\lvert\omega\times\ddot{\omega}\rvert$
$A_n = A_{n-1}\cdot\left[1+\frac{1}{2}\nabla\theta_n^*+\frac{1}{2}\left(1-A_{n-1}^2\right)\right]$	$-\frac{h^2}{12}\int_0^t A\cdot(\omega\times\dot{\omega})\cdot\tilde{A}\,dt' +$ $+\frac{h^2}{24}\int_0^t A\cdot(\omega^2\omega)\cdot\tilde{A}\,dt'$	$\frac{h^2}{12}\sqrt{\lvert\omega\times\dot{\omega}\rvert^2+\frac{1}{4}\omega^6}$
$A_n = A_{n-1}\cdot\left[1+\frac{1}{2}\nabla\theta_n^*-\frac{1}{3}\lvert\nabla\theta_n^*\rvert^2+\right.$ $\left.+\frac{1}{2}\left(1-A_{n-1}^2\right)\right]$	$-\frac{h^2}{12}\int_0^t A\cdot(\omega\times\omega)\cdot\tilde{A}\,dt' +$ $+\frac{h^2}{24}\int_0^t A\cdot(\omega^2\omega)\cdot\tilde{A}\,dt'$	$\frac{h^2}{12}\sqrt{\lvert\omega\times\dot{\omega}\rvert+\frac{1}{4}\omega^6}$

TABLE 4.5. (CONTINUED)

Numerical method	Accumulated position error $\delta\theta$	Rate of departure $\left\lvert\frac{d\,\delta\theta}{dt}\right\rvert$
$A_n = A_{n-1}\cdot\left[1+\frac{1}{2}\nabla\theta_n^* - \frac{1}{8}\lvert\nabla\theta_n^*\rvert^2 + \right.$ $+\frac{1}{24}(\nabla\theta_n^*\times\nabla^2\theta_n^*) - \frac{1}{48}\lvert\nabla\theta_n^*\rvert^2\nabla\theta_n^* +$ $\left.+\frac{1}{2}(1-A_{n-1}^2)\right]$	$-\frac{h^3}{24}\int_0^t A\cdot(\omega\times\ddot{\omega})\cdot\tilde{A}\,dt' +$ $+\frac{h^3}{96}\int_0^t A\cdot(\omega^2\dot{\omega})\cdot\tilde{A}\,dt'$	$\frac{h^3}{24}\left\lvert\omega\times\ddot{\omega}-\frac{1}{4}\omega^2\dot{\omega}\right\rvert$
$A_n = A_{n-1}\cdot\left[e^{\frac{1}{2}\nabla\theta_n^*}+\frac{1}{2}(1-A_{n-1}^2)\right]$	$-\frac{h^2}{12}\int_0^t A\cdot(\omega\times\dot{\omega})\cdot\tilde{A}\,dt'$	$\frac{h^2}{12}\lvert\omega\times\dot{\omega}\rvert$
$A_n = A_{n-1}\cdot\left\{e^{\frac{1}{2}\left[\nabla\theta_n^*+\frac{1}{12}(\nabla\theta_n^*\times\nabla^2\theta_n^*)\right]}+\right.$ $\left.+\frac{1}{2}(1-A_{n-1}^2)\right\}$	$-\frac{h^3}{24}\int_0^t A\cdot(\omega\times\ddot{\omega})\cdot\tilde{A}\,dt'$	$\frac{h^3}{24}\lvert\omega\times\ddot{\omega}\rvert$
$A_n = A_{n-1}+\frac{1}{2}A_{n-1}\cdot\nabla\theta_n^*$	$-\frac{h^2}{12}\int_0^t A\cdot(\omega\times\dot{\omega})\cdot\tilde{A}\,dt' -$ $-\frac{h^2}{12}\int_0^t A\cdot(\omega^2\omega)\cdot\tilde{A}\,dt'$	$\frac{h^2}{12}\sqrt{\lvert\omega\times\dot{\omega}\rvert^2+\omega^6}$
$A_n = A_{n-1}+\frac{1}{2}A_{n-1}\cdot\nabla\theta_n^*+\frac{1}{4}\nabla A_n\cdot\nabla\theta_n^*$ $\nabla A_n = \nabla A_{n-1}+\nabla^2 A_{n-1}+\nabla^3 A_{n-1}$ $\cdots$	$-\frac{h^2}{12}\int_0^t A\cdot(\omega\times\dot{\omega})\cdot\tilde{A}\,dt' +$ $+\frac{h^2}{48}\int_0^t A\cdot(\omega^2\omega)\cdot\tilde{A}\,dt'$	$\frac{h^2}{12}\sqrt{\lvert\omega\times\dot{\omega}\rvert^2+\frac{1}{16}\omega^6}$
$A_n = A_{n-1}+\frac{1}{2}A_{n-1}\cdot\nabla\theta_n^*+\frac{1}{4}\nabla A_n\cdot\nabla\theta_n^*+$ $+\frac{1}{24}(\nabla A_n\cdot\nabla^2\theta_n^*-\nabla^2 A_n\cdot\nabla\theta_n^*)$ $A_n = \nabla A_{n-1}+\nabla^2 A_{n-1}+\nabla^3 A_{n-1}+\nabla^4 A_{n-1}$ $\nabla^2 A_n = \nabla^2 A_{n-1}+\nabla^3 A_{n-1}+\nabla^4 A_{n-1}$	$-\frac{h^3}{24}\int_0^t A\cdot(\omega\times\ddot{\omega})\cdot\tilde{A}\,dt' +$ $+\frac{h^3}{48}\int_0^t A\cdot(\omega^2\dot{\omega})\cdot\tilde{A}\,dt'$	$\frac{h^3}{24}\left\lvert\omega\times\ddot{\omega}-\frac{1}{2}\omega^2\dot{\omega}\right\rvert$

By converting (4.59) with the help of (4.53), (4.54), (4.56), we obtain:

$$\delta\Lambda_n^{(n)} = \Lambda_0 \circ [N_1 + \delta N_1^{(n)}] \circ \ldots \circ [N_n + \delta N_n^{(n)}] \circ \tilde{N}_n \circ \ldots \circ \tilde{N}_1 \circ \tilde{\Lambda}_0.$$

Here:

$$\delta N_n^{(n)} = \frac{1}{N_n}(N_n + \delta N_n) - N_n$$

is the error per increment remaining after normalization. The explicit form of $\delta N_n^{(n)}$ for the various numerical methods is presented in Table 4.4. We note that it is this component which leads to the error in constructing the coordinate systems.

Then, we have by analogy with the preceding case:

$$\delta\Lambda^{(n)} = 1 + \sum_{k=1}^{n} \Lambda_0 \circ N_1 \circ \ldots$$
$$\ldots \circ N_{k-1} \circ \delta N_k^{(n)} \circ \tilde{N}_{k-1} \circ \ldots \circ \tilde{N}_1 \circ \tilde{\Lambda}_0 + O(\delta N_k^{*2}) =$$
$$= 1 + \sum_{k=1}^{n} \Lambda_{k-1} \circ \delta N_k^{(n)} \circ \tilde{\Lambda}_k + O(\delta N_k^{*2}).$$

By replacing the sum by the integral, we obtain: / 169

$$\delta\Lambda^{(n)} \approx 1 + \frac{1}{h}\int_0^t \Lambda(t') \circ \delta N^{(n)}(t') \circ \tilde{\Lambda}(t')\, dt'.$$

We represent $\delta\Lambda^{(n)}$ with the true rotation vector $\boldsymbol{\theta}$.

$$\delta\Lambda^{(n)} = \cos\frac{1}{2}\delta\theta + \zeta \sin\frac{1}{2}\delta\theta.$$

Hence, we find:

$$\delta\theta = 2^{*}_{\circ} \operatorname{arctg} \frac{|\operatorname{vect}[\delta\Lambda^{(n)}]|}{\operatorname{sqal}(\delta\Lambda^{(n)})}$$

or

$$\delta\theta \approx 2 \operatorname{vect}[\delta\Lambda^{(n)}].$$

The quantity $\delta\Theta$ determines the deviation of the true position of the coordinate system from the calculated, i.e., the position error.

It is easy to see that:

$$\operatorname{vect}\left(\int_0^t \Lambda \circ \delta N^{(n)} \circ \tilde{\Lambda}\, dt'\right) = \int_0^t \Lambda \circ \operatorname{vect}(\delta N^{(n)}) \circ \tilde{\Lambda}\, dt'.$$

With consideration of this, we obtain the following expression for the accumulated error of the true rotation ("angle") vector:

$$\delta\Theta \approx \frac{2}{h}\int_0^t \Lambda(t') \circ \operatorname{vect}[\delta N^{(n)}(t')] \circ \tilde{\Lambda}(t')\, dt'. \qquad (4.60)$$

The explicit expressions for $\delta\Theta$ and the rates of its departure for the various numerical methods are presented in Table 4.5.

We note that the order of magnitude of $\delta\Theta$ is not less than h^2 for all methods, including the Euler method.

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§ 4.5. NUMERICAL METHODS OF INTEGRATION WITH VARIABLE INCREMENTS

The input of information about the angular velocity into a computer requires the representation of this information in quantized form. In platformless systems, as a rule, the process of quantizing the angular velocity more precisely, the increments of the vector $\Theta^{\bullet}$ coincides with the measurement process. In other words, the information at the output of the measuring unit (sensors) of the angular velocity is represented in the form of a quantized flow of increments $\nabla\hat{\Theta}^{\bullet\bullet}$ (quantized quantities will be denoted by two asterisks). Then, the elementary

increment (pulse) in a given channel * has the form:

$$\Delta\theta_1^{**} = \pm\varepsilon e_1,\quad \Delta\theta_2^{**} = \pm\varepsilon e_2,\quad \Delta\theta_3^{**} = \pm\varepsilon e_3$$

where ε is the magnitude of the quantum.

To execute the algorithms of the numerical integration with constant increment, it is necessary to accumulate the elementary increments at the computer input in special registers and to interrogate these registers periodically (at the end of the following increment).

For a sufficiently fast computer response, one can exclude the process of increment accumulation and take the end of the following increment of integration from the instant of admitting the following elementary increment from any of the channels. Then, of course, the magnitude of the integration increment will become a variable depending on the rate of admission of the elementary increments at the computer input.

Under these conditions, one can apply directly the numerical methods of integration using only the first differences, i.e., the ordinary and modified Euler methods (4.11) and (4.12), the average velocity method (4.17) and the corresponding methods with norm correction. The formulae for evaluating the error derived above are not directly applicable in this case.

We introduce the total rotation angle: /177

$$\Phi = \int_0^t \omega(t')\,dt'$$

as an independent variable in the kinematic equations. As was shown in paragraph 3.4, the form of the kinematic equations is

* We will assume that the channel contains a sensor and a transforming device which measure the projection of the angular velocity onto a given axis.

maintained with such a replacement, but the role of the angular velocity vector (in our case ω_ε) is performed by its basis vectors:

$$e_\omega = (e_{\omega_1}, e_{\omega_2}, e_{\omega_3}).$$

Thus, by denoting differentiation by Φ with the prime, we have in accordance with (3.74):

$$\Lambda' = \frac{1}{2}\Lambda \circ e_\omega.$$

The form of all the formulae of this chapter will be maintained in exactly the same way, except that in place of the increment in time **h**, the increment in total rotation angle **g** will appear in the corresponding places.

Thus, the replacement of the argument permits one to extend the formulae of the preceding paragraphs to the case of variable h, but constant **g**. Then:

$$\nabla^k \Theta^\cdot \doteq g^k \frac{d^{k-1}}{dt^{k-1}}(e_\omega).$$

When operating from each elementary increment, the increment g, generally speaking, does not remain constant, but varies within sufficiently narrow limits:

$$\varepsilon \leqslant g \leqslant \frac{1}{\sqrt{3}}\varepsilon.$$

Let us consider the average value of the increment **g** over some integration interval. Let the total number of pulses (elementary increments) in the i-th channel equal K_i. We set:

$$g_{ave} = \frac{\Phi}{K_1 + K_2 + K_3}.$$

On the other hand:

$$K_i \approx \frac{1}{\varepsilon}\int_0^t |\omega_i| d\tau = \frac{1}{\varepsilon}\int_0^\Phi |e_{\omega_i}| d\Phi' \approx \frac{1}{\varepsilon}|e_{\omega_i}|\Phi.$$

Then: 178

$$g_{ave} = \frac{\varepsilon}{|e_{\omega_1}| + |e_{\omega_2}| + |e_{\omega_3}|}$$

or finally:

$$g_{ave} = \chi\varepsilon,$$

where:

$$\chi \approx \frac{1}{|e_{\omega_1}| + |e_{\omega_2}| + |e_{\omega_3}|}.$$

Obviously:

$$\frac{1}{\sqrt{3}} \leqslant \chi \leqslant 1.$$

It is natural to assume that χ and, consequently, g_{ave} are slowly changing variables. In this case, by analogy with (4.58) and (4.60), one can obtain an estimate of the accumulated error for operation from the elementary increments:

$$\left.\begin{aligned} \delta\Lambda^* &= \int_0^{\Phi} \delta N \frac{d\Phi'}{g}, \\ \delta\Theta &= 2\int_0^{\Phi} \Lambda \circ \mathrm{vect}(\delta N^{(n)}) \circ \tilde{\Lambda} \frac{d\Phi'}{g}. \end{aligned}\right\} \qquad (4.61)$$

The explicit expressions for $\delta\Lambda^*$, $\delta\Theta$ and the respective rates of departure for the various numerical methods are presented in the Tables 4.6 and 4.7.

By analyzing the integration formulae (4.11), (4.12), (4.17) and the corresponding integration formulae with norm correction, one can show that for operation from the increments, their execution leads to a comparatively small volume of calculations per increment. For the ordinary and modified Euler methods, one should select for this the magnitude of the quantum equal to 2^{-k}, where k — is a whole number. In that case, the multiplication operations are eliminated or replaced by displacements. For the average velocity method, it is advantageous, for the same reasons, to take:

$$\varepsilon = 2\arcsin 2^{-k},$$

TABLE 4.6. ACCUMULATED MAGNITUDE ERROR WITH OPERATION FROM ELEMENTARY INCREMENTS AND RATE OF DEPARTURE /179

Numerical method	Accumulated error $\delta\Lambda^*$	Rate of departure $\frac{d\,\delta\Lambda^*}{dt}$
$\Lambda_n = \Lambda_{n-1} \cdot \left(1 + \frac{1}{2} \nabla\Theta_n^{**}\right)$	$\frac{1}{8} \varepsilon \int_0^{\Phi} \chi \, d\Phi$	$\frac{1}{8} \varepsilon \chi \omega$
$\Lambda_n = \Lambda_{n-1} \cdot \left(1 + \frac{1}{2} \nabla\Theta_n^{**} - \frac{1}{8} \varepsilon^2\right)$	$\frac{1}{128} \varepsilon^3 \int_0^{\Phi} \chi^3 \, d\Phi$	$\frac{1}{128} \varepsilon^3 \chi^3 \omega$
$\Lambda_n = \Lambda_{n-1} \cdot e^{\frac{1}{2} \nabla\Theta_n^{**}}$	—	—
$\Lambda_n = \Lambda_{n-1} \cdot \left[1 + \frac{1}{2} \nabla\Theta_n^{**} + \frac{1}{2}\left(1 - \Lambda_{n-1}^2\right)\right]$	$\frac{1}{8} \varepsilon^2$	—
$\Lambda_n = \Lambda_{n-1} \cdot \left[1 + \frac{1}{2} \nabla\Theta_n^{**} - \frac{1}{8} \varepsilon^2 - \frac{1}{2}\left(1 - \Lambda_{n-1}^2\right)\right]$	$\frac{1}{128} \varepsilon^4$	—
$\Lambda_n = \Lambda_{n-1} \cdot \left[e^{\frac{1}{2} \nabla\Theta_n^{**}} + \frac{1}{2}\left(1 - \Lambda_{n-1}^2\right)\right]$	—	—

TABLE 4.7. ACCUMULATED POSITION ERROR AND RATE OF DEPARTURE WITH OPERATION FROM ELEMENTARY INCREMENTS /180-1

Numerical method	Accumulated error $\delta\Theta$	Rate of departure $\left\lvert\frac{d\,\delta\Theta}{dt}\right\rvert$
$A_n = A_{n-1} \cdot \left(1 + \frac{1}{2} V\Theta_n''\right)$	$-\frac{\varepsilon^2}{12}\int_0^{\Phi} A \cdot \left(e_\omega \times \frac{de_\omega}{d\Phi}\right) \cdot \tilde{A}\chi^2\, d\Phi' - -\frac{\varepsilon^2}{12}\int_0^{\Phi} A \cdot e_\omega \cdot \tilde{A}\chi^2\, d\Phi'$	$\frac{\varepsilon^2}{12}\chi^2\omega\sqrt{1+\left\lvert\frac{de_\omega}{d\Phi}\right\rvert^2}$
$A_n = A_{n-1} \cdot \left(1 + \frac{1}{2} V\Theta_n'' - \frac{1}{8}\varepsilon^2\right)$	$-\frac{\varepsilon^2}{12}\int_0^{\Phi} A \cdot \left(e_\omega \times \frac{de_\omega}{d\Phi}\right) \cdot \tilde{A}\chi^2\, d\Phi' + +\frac{\varepsilon^2}{24}\int_0^{\Phi} A \cdot e_\omega \cdot \tilde{A}\chi^2\, d\Phi'$	$\frac{\varepsilon^2}{12}\chi^2\omega\sqrt{\frac{1}{4}+\left\lvert\frac{de_\omega}{d\Phi}\right\rvert^2}$
$A_n = A_{n-1} \cdot e^{\frac{1}{2} V\Theta_n''}$	$-\frac{\varepsilon^2}{12}\int_0^{\Phi} A \cdot \left(e_\omega \times \frac{de_\omega}{d\Phi}\right) \cdot \tilde{A}\chi^2\, d\Phi'$	$\frac{\varepsilon^2}{12}\chi^2\omega\left\lvert\frac{de_\omega}{d\Phi}\right\rvert$
$A_n = A_{n-1} \cdot \left[1 + \frac{1}{2} V\Theta_n'' + \frac{1}{2}\left(1 - A_{n-1}^2\right)\right]$	$-\frac{\varepsilon^2}{12}\int_0^{\Phi} A \cdot \left(e_\omega \times \frac{de_\omega}{d\Phi}\right) \cdot \tilde{A}\chi^2\, d\Phi' + +\frac{\varepsilon^2}{24}\int_0^{\Phi} A \cdot e_\omega \cdot \tilde{A}\chi^2\, d\Phi'$	$\frac{\varepsilon^2}{12}\chi^2\omega\sqrt{\frac{1}{4}+\left\lvert\frac{de_\omega}{d\Phi}\right\rvert^2}$
$A_n = A_{n-1} \cdot \left[1 + \frac{1}{2} V\Theta_n'' - \frac{1}{8}\varepsilon^3 + + \frac{1}{2}\left(1 - A_{n-1}^2\right)\right]$	$-\frac{\varepsilon^2}{12}\int_0^{\Phi} A \cdot \left(e_\omega \times \frac{de_\omega}{d\Phi}\right) \cdot \tilde{A}\chi^2\, d\Phi' + +\frac{\varepsilon^2}{24}\int_0^{\Phi} A \cdot e_\omega \cdot \tilde{A}\chi^2\, d\Phi'$	$\frac{\varepsilon^2}{12}\chi^2\omega\sqrt{\frac{1}{4}+\left\lvert\frac{de_\omega}{d\Phi}\right\rvert^2}$
$A_n = A_{n-1} \cdot \left[e^{\frac{1}{2} V\Theta_n''} + \frac{1}{2}\left(1 - A_{n-1}^2\right)\right]$	$-\frac{\varepsilon^2}{12}\int_0^{\Phi} A \cdot \left(e_\omega \times \frac{de_\omega}{d\Phi}\right) \cdot \tilde{A}\chi^2\, d\Phi'$	$\frac{1}{12}\chi^2\omega\left\lvert\frac{de_\omega}{d\Phi}\right\rvert$

On the other hand, the use of registers at the input of the integrating device permits one to apply the numerical methods of high order with a comparatively large increment. The average number of operations per second for the same error can then be even less than for operation with the elementary increments. Thus, the selection of the integration method can be made only on the basis of a detailed analysis of the specific problem.

One should also keep in mind that under real conditions, the information being admitted to the input of the computer can contain noise. Thus, one should avoid algorithms, which directly or indirectly use the differentiation operation. In this sense the algorithms (4.11) - (4.13) and (4.17) - (4.19) using the approximate solution of the kinematic equation have definite advantages over the classical difference methods, including the formulae (4.22) - (4.25). In addition, the function $\omega_{\varepsilon}(t)$ itself will not always be sufficiently smooth and, in this case, the use of the methods of high order can even increase the integration error.

§ 4.6. QUANTIZATION ERROR

The specific error caused by the fact that the quantized quantity $\nabla\boldsymbol{\theta}^{*}$ is used in place of the true value of the vector $\nabla\boldsymbol{\theta}^{**}$ was not considered in deriving the formulae of the preceding paragraphs. The quantization process occurs outside the computer. Thus, the quantization error does not depend either on the integration increment or on the numerical method applied, and it results from the noncompucativity of finite rotation

addition *.

To evaluate the quantization error, let us consider the plane rotation specified by the basis vector e_ω.

For the plane rotation, the vectors θ, θ^* and $e_\omega\Phi$ coincide, thus:

$$\Lambda = \Lambda_0 \circ e^{\frac{1}{2} e_\omega \Phi}.$$

We denote the total rotation angle per interval between neighboring pulses in the i-th channel by Φ_i. Obviously:

$$\Phi_i = \frac{\varepsilon}{|e_{\omega_i}|} \qquad (i = 1, 2, 3).$$

To be definite, let:

$$\Phi_1 \geqslant \Phi_2 \geqslant \Phi_3 \tag{4.62}$$

Then, obviously, the number of pulses K_i in the i-th channel per interval Φ satisfies the condition:

$$K_1 \leqslant K_2 \leqslant K_3.$$

Let us select the interval Φ equaling Φ_1. Then $K_1 = 1$; in addition, on the average **over several intervals:

$$K_2 = \frac{|e_{\omega_2}|}{|e_{\omega_1}|}, \quad K_3 = \frac{|e_{\omega_3}|}{|e_{\omega_1}|}.$$

---

* An additional error component caused by rounding off arises in the process of calculation in digital computers with a finite register length. Its magnitude depends on the increment length, the number of operations per increment, the rounding-off method, the number of registers, and also whether the calculations are being performed in the fixed point or in the floating point mode. This component will not be considered in this paragraph.

* The quantities K_2, K_3, K_{23} considered as averages over some interval cannot, be generally speaking, whole numbers.

Let us consider the average number of pulses in the third channel between neighboring pulses in the second channel:

$$K_{23} = \frac{|e_{\omega_2}|}{|e_{\omega_3}|}.$$

To evaluate the error, one should specify the most unfavorable distribution of pulses per interval Φ. We assume that K_{23} pulses are first admitted in the third channel, then one pulse in the second channel. This process is repeated K_2 times, after which one pulse is admitted in the first channel. Such a sequence of pulses (elementary increments) corresponds to the quaternion:

$$\Lambda^{***} = \Lambda_0 \circ e^{\frac{1}{2} i_3 \varepsilon K_{23} \operatorname{sign}(e_{\omega_3})} \circ e^{\frac{1}{2} i_2 \varepsilon \operatorname{sign}(e_{\omega_2})} \circ e^{\frac{1}{2} i_1 \varepsilon \operatorname{sign}(e_{\omega_1})}.$$

At the same time, the true motion corresponds to the quaternion: /184

$$\Lambda = \Lambda_0 \circ e^{\frac{1}{2} e_{\omega_1} \frac{\varepsilon}{|e_{\omega_1}|}}.$$

the mismatch quaternion equals:

$$\Lambda_p = \Lambda^{***} \circ \tilde{\Lambda}.$$

Assuming that:

$$\frac{\varepsilon}{|e_{\omega_1}|} \ll 1, \tag{4.63}$$

after the corresponding transformations, we obtain:

$$\Lambda_p = 1 + \frac{1}{4} \varepsilon \Phi \{ i_1 e_{\omega_1} \operatorname{sign}(e_{\omega_1}) + i_2 e_{\omega_2} \operatorname{sign}(e_{\omega_2}) - i_3 e_{\omega_3} \operatorname{sign}(e_{\omega_3}) + \ldots$$

From (4.61), we have:

$$\left| \frac{d\delta\theta}{d\Phi} \right| = 2 \left| \operatorname{vect} \left(\frac{d\Lambda_p}{d\Phi} \right) \right|$$

or:

$$\left| \frac{d\delta\theta}{d\Phi} \right| \leq \frac{\varepsilon}{2} \sqrt{2e_{\omega_2}^2 + e_{\omega_3}^2}.$$

Analogous formulae can be obtained, not only under the condition (4.62), but also for any relation between Φ_1, Φ_2, Φ_3. The difference will be only in the number of the corresponding indices.

The quantities e_{ω_i} are the direction cosines, thus:

$$e_{\omega_1}^2 + e_{\omega_2}^2 + e_{\omega_3}^2 = 1$$

and:

$$\left|\frac{d\delta\Theta}{d\mathbb{C}}\right| \leqslant \frac{\varepsilon}{\sqrt{2}}. \tag{4.64}$$

One can show that the estimate (4.64) remains valid even when the condition (4.63) used for its derivation is violated. It follows from formula (4.64) that to decrease the accumulated error in the position, it is necessary first to decrease the magnitude of the quantum. /185

§ 4.7. STABILIZATION OF THE MAGNITUDE AND ANGLE IN THE POISSON EQUATIONS.

As is known, the Poisson equations (3.38) can be written in vector form. For example, for the left group of Equations (3.38), we have:

$$\dot{a}_i = a_i \times \omega \quad (i = 1, 2, 3), \tag{4.65}$$

where:

$$a_i = (a_{1i}, a_{2i}, a_{3i}). \quad \omega = (\omega_1, \omega_2, \omega_3).$$

By analogy with paragraph 4.2, let us consider the equation equivalent to (4.65) for the unnormalized vector. The position of the body relative to some axis will be of interest, whereas the rotation about it will not. In this case one can be limited to the one Equation (4.65):

$$\dot{a} = a \times \omega.$$

It is easy to see that it is equivalent to the equation:

$$\dot{\mathbf{a}}=\frac{\dot{a}}{a}\mathbf{a}+\mathbf{a}\times\boldsymbol{\omega}$$

in the same sense as (4.31) and (4.32). By analogy with (4.34) for stabilizing the magnitude, one can set:

$$\dot{\mathbf{a}}=\mathbf{a}f(a)+\mathbf{a}\times\boldsymbol{\omega},$$

where, for example:

$$f(a)=-k(a^2-1).$$

In general, one should specify the position of the body relative to two noncoinciding axes:

$$\left.\begin{aligned}\dot{\mathbf{a}}&=\mathbf{a}\times\boldsymbol{\omega},\\ \dot{\mathbf{b}}&=\mathbf{b}\times\boldsymbol{\omega}.\end{aligned}\right\} \tag{4.66}$$

The equivalence is maintained and, for stabilization of the magnitudes, one can replace (4.66) by the equations:

$$\left.\begin{aligned}\dot{\mathbf{a}}&=\mathbf{a}f(a)+\mathbf{a}\times\boldsymbol{\omega},\\ \dot{\mathbf{b}}&=\mathbf{b}f(b)+\mathbf{b}\times\boldsymbol{\omega}.\end{aligned}\right\} \tag{4.67}$$

At the same time as the stabilization of the magnitudes of the vectors appearing in the Poisson equation, it is necessary /186 to stabilize the magnitude of the angle between them. Actually, by taking the scalar product of the first equation of (4.66) with **b**, the second — with **a** and adding, we have:

$$\frac{d(\mathbf{a}\cdot\mathbf{b})}{dt}=0.$$

For Equations (4.67), we obtain accordingly:

$$\frac{\dot{}}{dt}(\mathbf{a}\cdot\mathbf{b})=(\mathbf{a}\cdot\mathbf{b})[f(a)+f(b)].$$

We denoted $w=\cos(\widehat{\mathbf{a},\mathbf{b}})$. Then:

$$\mathbf{a}\cdot\mathbf{b}=abw.$$

Let us consider the derivative of w with respect to time. In view of (4.67):

$$\frac{dw}{dt}=\frac{d}{dt}\left[\frac{(a\cdot b)}{ab}\right]=\frac{ab\frac{d}{dt}(a\cdot b)-(a\cdot b)\frac{d}{dt}(ab)}{a^2b^2}=0.$$

Thus, Equations (4.67) have the first integral:

$$w(t)=w_0.$$

Unfortunately, with a deviation of w from the given value, the equivalence, in the sense as we have stated above, does not occur. Actually, for $w\neq w_0$ it is not clear which of the vectors deviates from the initial direction and in what direction. Thus, it is not clear without additional assumptions how the position of the body is to be reconstructed in this case.

In spite of this fact, one can construct stable control of the quantity w which maintains $w\approx w_0$. Then, one should take care that the process stabilizing w and the magnitudes proceed as independently as possible. To eliminate the indeterminacy, which was the topic of discussion above, one can require, for example, that in the control process the vectors **a** and **b** move in opposition along the shortest path with the same velocity.

With this taken into account, we proceed to construction of the control law. The magnitude stabilization process is not violated, if the structure of the Equations (4.67) is maintained. Thus, we set: /187

$$\left.\begin{aligned}\dot{a}&=af(a)+a\times(\omega+\omega_\kappa),\\ \dot{b}&=bf(b)+b\times(\omega-\omega_\kappa),\end{aligned}\right\}\qquad(4.68)$$

where ω_κ is the angular velocity of the correction. For $\omega_\kappa=0$ Equation (4.68) coincides with (4.67). By taking the scalar product of the first equation of (4.68) with **a**, and the second — with **b**, we have:

$$\left.\begin{aligned}\frac{da}{dt} &= af(a),\\ \frac{db}{dt} &= bf(b).\end{aligned}\right\} \tag{4.69}$$

By analogy:

$$\frac{dw}{dt} = -2\frac{(ab\omega_x)}{ab},$$

where $(ab\omega_x)$ denotes the scalar triple product of the vectors a, b, ω_x .

In order that the motion of the vectors **a** and **b** proceed along the shortest path, we set:

$$\omega_x = A(a \times b). \tag{4.70}$$

As a proportionality coefficient, we take an odd function of the mismatch $(w - w_0)$, for example:

$$A = \frac{k}{ab}(w - w_0). \tag{4.71}$$

With consideration of (4.70) and (4.71), we have after some transformations:

$$\frac{dw}{dt} = -2k(1 - w^2)(w - w_0). \tag{4.72}$$

Equations (4.68) take the form:

$$\left.\begin{aligned}\dot{a} &= af(a) + a \times \omega - \frac{k}{ab}[ba^2 - a(a \cdot b)](w - w_0),\\ \dot{b} &= bf(b) + b \times \omega - \frac{k}{ab}[ab^2 - b(a \cdot b)](w - w_0).\end{aligned}\right\} \tag{4.73}$$

Obviously, (4.69) and (4.72) are independent.

To prove the stability, we take the Lyapunov function in the form: /188

$$W = \alpha(a - 1)^2 + \beta(b - 1)^2 + \gamma(w - w_0)^2,$$
$$\alpha > 0, \quad \beta > 0, \quad \gamma > 0.$$

In accordance with Equations (4.73) or, the same thing, (4.69) and (4.72), we have:

$$\frac{dV}{dt} = 2\alpha a(a-1)f(a) + \\ + 2\beta b(b-1)f(b) - 4k\gamma(1-w^2)(w-w_0)^2.$$

Since $a > 0$ and $b > 0$, it is sufficient to require for stability that:

$$(a-1)f(a) < 0 \text{ for } a \neq 1,$$
$$(b-1)f(b) < 0 \text{ for } b \neq 1,$$
$$k > 0, \quad w^2 \neq 1.$$

For practical purposes it is sufficient that the equations be independent only in first approximation in magnitude and angle. Thus, the Equations (4.73) can be somewhat simplified. First, one can set $a \approx b \approx 1$ and $a \cdot b \approx w_0$ in the brackets in the expression. Then:

$$\left.\begin{aligned} \dot{a} &= af(a) + a \times \omega - k(w - w_0)(b - aw_0), \\ \dot{b} &= bf(b) + b \times \omega - k(w - w_0)(a - bw_0). \end{aligned}\right\} \qquad (4.74)$$

Further, one can replace $(w - w_0)$ approximately with $(a \cdot b - w_0)$, so that:

$$\left.\begin{aligned} \dot{a} &= af(a) + a \times \omega - k(a \cdot b - w_0)(b - aw_0), \\ \dot{b} &= bf(b) + b \times \omega - k(a \cdot b - w_0)(a - bw_0). \end{aligned}\right\} \qquad (4.75)$$

In the important special case when the vectors a and b must be mutually perpendicular, i.e., when $w_0 = 0$, Equations (4.75) take the simpler form:

$$\left.\begin{aligned} \dot{a} &= af(a) + a \times \omega - k(a \cdot b)b, \\ \dot{b} &= bf(b) + b \times \omega - k(a \cdot b)a. \end{aligned}\right\} \qquad (4.76)$$

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One can show that the simplified versions of the Equations (4.74) - (4.76) are stable, although the region of stability is narrower than for (4.73).

Equations (4.73) - (4.76) are more complicated than, for example, (4.36). Nonetheless, a quite efficient algorithm for

numerical integration can be constructed for them. The use of the Poisson equations with stabilization of the magnitude and angle is particularly convenient when a specific physical meaning can be attributed to the vectors **a** and **b**, for example, when the orientation of rigid body is investigated relative to the directions toward two given stars.

CHAPTER 5

USE OF QUATERNIONS IN PROBLEMS OF RIGID BODY MOTION CONTROL

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The kinematic parameters are generalized coordinates permitting one to describe the angular motion of a rigid body. On the other hand, the kinematic parameters have found application in problems of angular position control of a rigid body during its motion. The signal of the angular deviation of the rigid body from a given position in the control system is a function of the kinematic parameters. We will consider, in general form, an automatic control system of rigid body motion about its center of mass. The system should consist of angular position sensors, which determine the angular position (orientation) of the body relative to a given coordinate system and supply the angular deviation signal, a functional convector, which shapes the controlling signal from the angular deviation signal, and finally, a controlling device, with the help of which the angular position of the rigid body can be changed. The described control system forms the ordinary closed circuit of automatic control; angular position control systems for spacecraft, aircraft, rockets, ships, etc. are constructed in just this form.

We will consider the question of obtaining the angular deviation signals for a rigid body relative to a given coordinate system (reference basis).

Traditionally, functions of the Krylov angles or of the Euler angles have been most frequently used as these signals. This is related to the fact that, first of all, one has had to

use gyroscopic devices, which have Cardan suspensions, in automatic angular position control systems in a different way. Such devices permit one to measure the rotation angles of the axes of the Cardan rings, which are the Euler and Krylov angles.

The development of control technology and, in particular space technology has led to the appearance of angular position sentors of non-gyroscopic types. Examples of such sensors are /191 optical sensors for orientation toward the centers of planets or the sun, stellar orientation sensors, etc. The angular mismatch signals determined by these sensors are functions of the direction cosines, and not the Euler angles. When computers are used in the control system and the determination of the angular position of the body is carried out by integrating the kinematic equations, as occurs in platformless control systems, any kinematic parameters can be used as the angular mismatch signals. The form of the kinematic parameters in this case will be determined by the type of kinematic equations being integrated.

Thus, it is possible in principle to use various kinematic parameters in control problems. The use of quaternions appears to be the most convenient for two reasons. Firstly, the investigation of the stability of the orientation processes performed with the use of Rodriguez-Hamilton parameters permits one to judge the stability of the system as a whole. Quaternions permit one to find the most general form of the functional dependences ensuring stable control. Secondly, the quaternion components can be used as controlling signals, which can take place when these parameters are calculated in the control systems. The use of the quaternion components as control signals permits one to obtain not only stable control of the angular motion of a rigid body, but also, in a number of cases of control, they are sufficiently close to the optimum (cf. chapter 6).

By the problem of rigid body orientation in some given reference coordinate system I, we will understand the problem of reducing the body coordinate system E to the reference system. We will consider two formulations of the orientation problem; kinematic and dynamic.

In the first case we will assume that the orientation process includes imparting to the coordinate system E the absolute correction angular velocity $\Omega_к$, whose purpose is to change the /192
orientation of the basis E in such a fashion as to cause its coincidence with the coordinate system I. The process of reducing the basis E to the basis I, which occurs because of the presence of the angular velocity $\Omega_к$, will be called, by convention, the correction process. Such a formulation will be called the kinematic orientation problem. In practice it is very close to the problem of orientation and control of the angular position of a rigid body. Actually, an orientation control system, for example, of a spacecraft, controls the magnitude of its angular velocity which is specified as a function of the angular deviation of the craft from the required position. In spite of the fact that control is actually accomplished by applying controlling torques to the body (from rocket engines or by changing the angular momentum of internal masses), the nature of the motion with rather large accelerations from the controlling torque is close to that which will be considered in the kinematic orientation problem.

In the second case, the dynamic orientation problem, it will be assumed that the magnitude of the controlling torque M applied to the rigid body is the control, and not the correction angular velocity $\Omega_к$. The controlling torque causes a corresponding motion of the rigid body; the purpose of the orientation is still the coincidence of the body basis with the reference.

Types of control, i.e., the form of the functional dependences of Ω_K and **M** ensuring the performance of the orientation process, will be investigated in this chapter. The stability of the control process will be investigated; it will be shown that for some types of control the kinematic conditions for stability are necessary conditions for dynamic stability as well.

§ 5.1. DEPENDENCE OF THE CONTROL SIGNALS ON THE KINEMATIC PARAMETERS

A necessary condition for solving the orientation problem is to obtain control signals characterizing the angular mismatch of the rigid body coordinate system **E** and the reference system **I**. Actually, the control of rigid body orientation is based on /193
measurements of several angular position sensors, which determine the position of the body axes relative to certain directions in space (the direction toward a planet, the Sun, a star, spacecraft, etc.). We will assume for generality that the angular position sensors determine the position of the axes $\mathbf{i}_k$ of the basis **I** linked to the orientation directions.

We will distinguish two types of orientation: an orientation, in which the basis **I** is an inertial non-rotating coordinate system, called inertial orientation, and an orientation in a rotating coordinate system when the basis **I** performs the given rotation. In either case the output signals of the sensors can determine one, two or three angular coordinates of the basis mismatch.

We will assign to each output signal a direction for the sensor response axis in such a fashion that only this output signal arises with rotation about this axis from the coincidence position of the bases **E** and **I**. It is clear from general considerations that for spatial (triple-axis) orientation it is necessary

to have no less than three signals from sensors whose response axes do not lie in one plane (they form a spatial basis). The output signals of the angular position sensors can be represented as functions of the kinematic parameters characterizing the relative position of the bases.

We will consider various methods for representing the output signals of the angular position sensors.

Let the transition from the system **I** to the system **E** be determined with the help of some kinematic parameters and be represented by the matrix **A** expressed in terms of the Krylov angles (2.70) or quaternion components (1.19). Obviously, one set of parameters can be expressed in terms of the other set; however, we will consider only those parameters which are physically measurable quantities.

When the orientation is accomplished with the help of a three-stage gyrostabilized platform, the axial position of whose Cardan suspension corresponds to the selected sequence of Krylov angles (Figure 5.1), the output signals will be signals taken from angle gauges on the suspension axes. These signals will be direct functions of the Krylov angles ϑ, φ and ψ: $f_1(\vartheta), f_2(\varphi), f_3(\psi)$. Thus, for example, with the use of inductive angle gauges we will have: /194

$$f_1(\vartheta) = \sin\vartheta, \quad f_2(\varphi) = \sin\varphi, \quad f_3(\psi) = \sin\psi.$$

We note that the direction of each output signal of the gyroplatform f_k, i.e., the response axis, will be the corresponding axis of the Cardan suspension, and orientation with the help of the gyroplatform can be accomplished everywhere,

except when two axes of the Cardan suspension coincide.

As another example, let us consider a sensor made in the form of a sighting device in a double-stage Cardan suspension (Figure 5.2). The signals of angle gauges on the suspension axes can be the output signals of this sensor. If the sequence of angles for the transition from the sensor coordinate system to the object axes corresponds to the sequence considered above (cf. Figure 5.1, in which the third angle is taken equal to zero), then the measured angles will be φ and ψ. Accordingly, the response axes of such a sensor are arranged along the suspension axes.

One can consider the sensor fixed in the axes of the basis **E** and sighting in some direction, which can be the direction toward the planet, the Sun, a star, etc. In this more general case, the output signals of such a sensor will be functions of the direction cosines of the sighting line with the axes of the coordinate system **E**. Thus, if $\mathbf{i}_1$ is the sighting axis and the optical axis of the sensor is placed along the $\mathbf{e}_1$ axis, then the output signals will be the functions $f_1(e_2 \cdot i_1)$ and $f_2(e_3 \cdot i_1)$. /195
Obviously, the third cosine $e_1 \cdot i_1$ does not carry information about the direction of the deviation of the axis $\mathbf{e}_1$ from $\mathbf{i}_1$, being at the same time a measure of this deviation.

The position of the sensor relative to the axes of the coordinate system **E** is characterized by two of its response axes ξ_1 and ξ_2 (Figure 5.3). The signals f_1 and f_2 will arise with a deviation from the sighting direction by a rotation about the axes ξ_1 and ξ_2, respectively. A small angle deviation of the basis **E** from the given position, which is determined by the small rotation vector $\boldsymbol{\theta}$, can be partially measured by the sensor. The output signals of the sensor will depend on the quantities $\boldsymbol{\theta} \cdot \xi_1$ and $\boldsymbol{\theta} \cdot \xi_2$, which are the direction cosines of

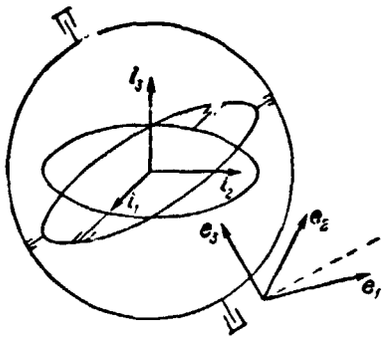

Figure 5.1.

the response axes ξ_1 and ξ_2 with the angular deviation vector. Obviously, such a two-coordinate sensor measures only the components of the angular deviation, which is not a rotation about the axis of the sighting line, and does not give complete information about the angular mismatch. The vector θ can be measured completely only when there are no less than three response axes not lying in one plane.

For three-axis orientation of a rigid body it is necessary to have information about the deviation of two axes, say $\mathbf{e}_1$ and $\mathbf{e}_2$, from the reference basis. If a fixed sensor, whose response axes /196 are arranged along the axes of the basis $\mathbf{E}$, is connected to each axis, then the output signals of this sensor will be functions of the direction cosines between the axes $\mathbf{i}_k$ and $\mathbf{e}_k$. By using formulae (1.19), we can write the following relations for the direction cosines:

$$\left.\begin{aligned} &\boldsymbol{i}_1\cdot\boldsymbol{e}_2=2(\lambda_1\lambda_2-\lambda_0\lambda_3), \quad \boldsymbol{i}_2\cdot\boldsymbol{e}_3=2(\lambda_3\lambda_2-\lambda_0\lambda_1),\\ &\boldsymbol{i}_1\cdot\boldsymbol{e}_3=2(\lambda_1\lambda_3+\lambda_0\lambda_2), \quad \boldsymbol{i}_3\cdot\boldsymbol{e}_1=2(\lambda_1\lambda_3-\lambda_0\lambda_2),\\ &\boldsymbol{i}_2\cdot\boldsymbol{e}_1=2(\lambda_0\lambda_3+\lambda_1\lambda_2), \quad \boldsymbol{i}_3\cdot\boldsymbol{e}_2=2(\lambda_3\lambda_2+\lambda_0\lambda_1). \end{aligned}\right\} \quad (5.1)$$

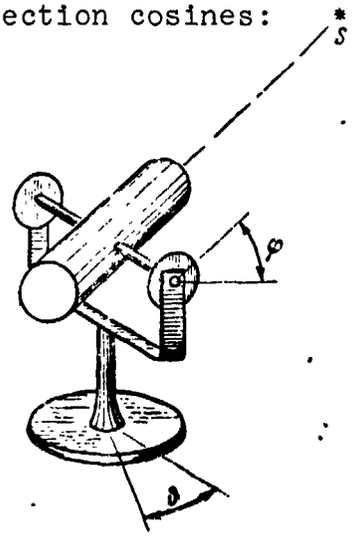

Figure 5.2.

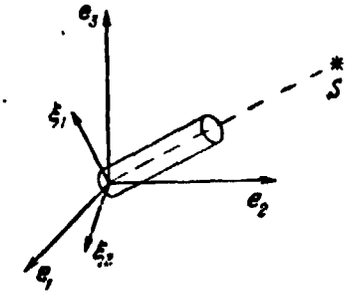

Figure 5.3.

In the case of small deviations of the basis E from the basis I, we obtain, by comparing (2.70), (1.19) and (2.1) with consideration of quantities of first order relative to the spatial rotation angle only:

$$\left.\begin{aligned} \vartheta &\approx e_1 \cdot i_2 \approx -e_2 \cdot i_1 \approx 2\lambda_0\lambda_3, \\ \varphi &\approx e_2 \cdot i_3 \approx -e_3 \cdot i_2 \approx 2\lambda_0\lambda_1, \\ \psi &\approx e_3 \cdot i_1 \approx -e_1 \cdot i_3 \approx 2\lambda_0\lambda_2. \end{aligned}\right\} \qquad (5.2)$$

These relations link the various kinematic parameters for the case of small deviations. In practice, they also have value for describing the functional dependences of the sensor output signals, since a majority of sensors have an output characteristic with a small "linear" region, in which the output signal carries information about the magnitude of the deviation angle. In the remaining "operating" region, as a rule, the sensor given information only about the sign of the deviation; the operating region will also be partially limited. In accordance with this, the output characteristic of a sensor can be represented as a function of the quantities (5.2).

The relations (5.2) indicate that the quantities $\lambda_0\lambda_i$ can be used for control along with the angles ϑ, φ and ψ traditionally used. A strict justification of this assertion will be given below. The control signals shaped on the basis of the output signals of the angular position sensors will be specified as functions of the quantities (5.1) or (5.2) in the following considerations.

§ 5.2. KINEMATIC PROBLEM OF ORIENTATION IN AN INERTIAL COORDINATE SYSTEM

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We will consider the kinematic problem of rigid body orientation in the basis I, an inertial coordinate system. Let the

motion body basis E relative to the basis I be described by the quaternion $\Lambda(t)$ and the absolute angular velocity of rotation of the basis E be equal to $\Omega_к$, the correction velocity generated from the angular position sensor signals.

We will investigate the process of reducing the system E to the inertial system I for various dependences of the correction signals on the kinematic parameters and will find the requirements on the control, which ensure stability of the correction system. The kinematic equations in the form (3.11) will be the equations of motion of the basis E relative to the basis I, since the correction velocity $\Omega_к$ is specified in the axes of the basis E by the control system, which controls the magnitudes of the angular velocity projections $\Omega_{кi}$ (i = 1,2,3) onto the body axes. We will write these equations in scalar form:

$$\left.\begin{aligned} 2\dot{\lambda}_0 &= -\lambda_1\Omega_{к1} - \lambda_2\Omega_{к2} - \lambda_3\Omega_{к3}, \\ 2\dot{\lambda}_1 &= \lambda_0\Omega_{к1} + \lambda_2\Omega_{к3} - \lambda_3\Omega_{к2}, \\ 2\dot{\lambda}_2 &= \lambda_0\Omega_{к2} + \lambda_3\Omega_{к1} - \lambda_1\Omega_{к3}, \\ 2\dot{\lambda}_3 &= \lambda_0\Omega_{к3} + \lambda_1\Omega_{к2} - \lambda_2\Omega_{к1}. \end{aligned}\right\} \qquad (5.3)$$

We will investigate the sufficient conditions for orientation stability and the form of the functional dependences of the quantities $\Omega_{кi}$, which ensure such orientation.

I. We introduce the positive-definite function:

$$W = 1 - \lambda_0^2 = \frac{1}{2}[(1-\lambda_0^2) + \lambda_1^2 + \lambda_2^2 + \lambda_3^2]. \qquad (5.4)$$

This function vanishes for the oriented position of the rigid body, i.e., when the bases E and I coincide (this position corresponds to the quaternion (1, 0, 0, 0)); for all other positions of the basis E relative to the reference basis I, it has a positive value. The derivative of this function, which is composed in accordance with the equations of motion (5.3), will

equal. /198

$$\dot{W} = -2\lambda_0\dot{\lambda}_0 = \lambda_0\lambda_1\Omega_{x1} + \lambda_0\lambda_2\Omega_{x2} + \lambda_0\lambda_3\Omega_{x3}. \qquad (5.5)$$

We will first consider the case when the quaternion components are used for control, i.e., the controlling signal is generated as a function of the quantities λ_j $(j = 1, 2, 3)$. This case can occur when the components λ_j are calculated in the control system.

The relations (5.2) show that the quantities $\lambda_0\dot{\lambda}_j$, which coincide with the corresponding angles for small deviations, serve as measures of the angular deviation. If one takes into account that two quaternions (Λ and $-\Lambda$) correspond to one position of the body, then one can see that the quantities $\lambda_0\dot{\lambda}_j$ carry information about the direction of the shorter spatial turn, whose magnitude is less than π.

In the case of linear correction, the magnitude of the correction angular velocity should be given in the form:

$$\Omega_{xj} = -k_j\lambda_0\lambda_j. \qquad (5.6)$$

One can see that for $k_j > 0$ the derivative (5.5) will be a negative-definite form everywhere, except for the oriented position:

$$-\dot{W} = k_1\lambda_0^2\lambda_1^2 + k_2\lambda_0^2\lambda_2^2 + k_3\lambda_0^2\lambda_3^2.$$

Thus, the function (5.4) is a Lyapunov function, and the Equation (5.6) ensures an asymptotic convergence of the correction process to the equilibrium position, i.e., to the position $\Lambda = (1, 0, 0, 0)$, over the whole region of variation of Λ, except for the point $\lambda_0 = 0$ (generally speaking, the point $\lambda_0 = 0$ corresponds to some region of positions).

As follows from the relation (5.6), the velocity Ω_{kj} also vanishes for $\lambda_0 = 0$; the derivative W also vanishes. This state is a position of unstable equilibrium. To determine the nature of this point, we set $k_j = k$; in this case:

$$\dot{W} = -k\lambda_0^2(1-\lambda_0^2) = -kW(1-W).$$

By integrating this equation, we obtain: /199

$$W = \frac{W_0 e^{-kt}}{1 + W_0(e^{-kt} - 1)},$$

where W_0 equals the value W at the initial time. Hence, it is seen that the rate of convergence depends on the correction coefficient k, and W(t), and W(t) tends to zero with $k > 0$ for the initial values $W_0 < 1$. Obviously, W cannot be one anywhere, except at the point $\lambda_0 = 0$; for $W_0 = 1$, we have W(t) - 1.

Let $W_0 = 1 - \delta$, where δ is an infinitesimal; then:

$$W(t) \approx \frac{1}{1 + \delta e^{kt}},$$

i.e., the singular point $\lambda_0 = 0$ is an unstable node.

Control by the quantities $\lambda_0 \lambda_j$ will be asymptotically stable for nonlinear correction functions as well.

Let the control have the form:

$$\Omega_{kj} = \pm k_j F_j(\lambda_0)\Phi_j(\lambda_j), \qquad (5.7)$$

where the functions $F_j(x)$ and $\Phi_j(x)$ satisfy the conditions:

$$xF_j(x) > 0, \quad x\Phi_j(x) > 0, \quad |x| \leqslant 1. \qquad (5.8)$$

We will henceforth say that Equation (5.7) belongs to the class (5.8). For such control, the derivative of the Lyapunov function (5.5) will be negative-definite for $k_j > 0$ everywhere, except for the singular point $\lambda_0 = 0$, at which it vanishes. Hence, it follows that

any type of nonlinear control by the linear components (5.7) belong-belonging to the class (5.8) ensures asymptotic stability of the control process everywhere that $\lambda_0 \neq 0$.

For asymptotic stability over the whole range of rigid body positions, it is sufficient to redefine the Equations (5.6) and (5.7) such that they do not vanish at the point $\lambda_0 = 0$. In this case, in spite of the fact that the derivative of the Lyapunov function vanishes at the point $\lambda_0 = 0$, this position will no longer be an equilibrium position (not even unstable), as in the example of linear control (5.6) considered previously. To prove this fact, we refer to the Lyapunov theorem [1] in the formulation: if there exists a positive-definite function of phase / 200
coordinates such that its derivative is not positive and vanishes on some set not containing entire trajectories besides the equilibrium position, then the equilibrium position of the system is asymptotically stable.

II. We will consider the case of control by direction cosines: we will investigate the equations, which are linear functions of the direction cosines (5.1).

Obviously, it is sufficient to use two pairs of cosines determining the orientation of two axes of the coordinate system I in the body basis to accomplish three-axis orientation; only three of the four cosines being independent. Let, for example, the cosines of the axes $\mathbf{i}_1$ and $\mathbf{i}_2$ be used. Then one can generate the three values of the correction angular velocity Ω_{ki} as linear functions of the cosines (1.5). By taking into account that the generated correction signals must take the form (5.6) in accordance with (5.2) for small deviations, we obtain:

$$\left.\begin{aligned}
\Omega_{x1} &= \frac{k_1}{2}(i_2 \cdot e_3) = -k_1(\lambda_1\lambda_0 - \lambda_3\lambda_2),\\
\Omega_{x2} &= -\frac{k_2}{2}(i_1 \cdot e_3) = -k_2(\lambda_2\lambda_0 + \lambda_1\lambda_3),\\
\Omega_{x3} &= -\frac{k_3}{2}(i_2 \cdot e_1) = -k_3(\lambda_3\lambda_0 + \lambda_1\lambda_2),\\
\Omega_{x3} &= \frac{k_3}{2}(i_1 \cdot e_2) = -k_3(\lambda_3\lambda_0 - \lambda_1\lambda_2),
\end{aligned}\right| \qquad (5.9)$$

i.e., there are two possibilities for generating three correction signals from four direction cosines. We will show that it is possible to satisfy the sufficient conditions for stability for both cases. For the proof we will find the Lyapunov function in the following fo.m:

$$W = \alpha(1 - \lambda_0^2) + \beta\lambda_3^2 + \gamma\lambda_2^2 + \delta\lambda_1^2. \qquad (5.10)$$

where α, β, γ and δ are arbitrary positive parameters. obviously, this function is positive-definite and vanishes only at the equilibrium position. /201

The derivative of this function composed in accordance with Equations (5.3) has the following form:

$$\begin{aligned}
-\frac{1}{2}\dot{W} = {} & k_1(\alpha+\delta)\lambda_0^2\lambda_1^2 + k_2(\alpha+\gamma)\lambda_0^2\lambda_2^2 + k_3(\alpha+\beta)\lambda_0^2\lambda_3^2 + \\
& + k_1(\beta-\gamma)\lambda_2^2\lambda_3^2 + k_2(\beta-\delta)\lambda_1^2\lambda_3^2 \pm k_3(\delta-\gamma)\lambda_1^2\lambda_2^2 + \\
& + [k_1(\gamma-\beta-\alpha-\delta) + k_2(\alpha+\gamma+\beta-\delta) + \\
& + k_3(\delta-\gamma\pm\alpha\pm\beta)]\lambda_0\lambda_1\lambda_2\lambda_3.
\end{aligned} \qquad (5.11)$$

where the upper sign corresponds to the third, and the lower to the fourth equality of (5.9), which defines the control Ω_{x3}. In order that the derivative (5.11) will be a negative function, it is necessary to satisfy the following conditions:

$$k_i > 0, \quad \beta \geqslant \gamma, \quad \beta \geqslant \delta, \quad \pm(\delta-\gamma) \geqslant 0 \quad (i = 1, 2, 3). \qquad (5.12)$$

We will show that the fixed sign of the function W over the whole region of variation of the quantity λ_i can be obtained by the selection of the parameters $\alpha, \beta, \gamma, \delta$. We represent the function (5.11) in the following form:

$$-\frac{1}{2}\dot{W} = k_1(\alpha+\delta)\lambda_0^2\lambda_1^2 + k_1(\beta-\gamma)\lambda_2^2\lambda_3^2 + a\lambda_0\lambda_1\lambda_2\lambda_3 + \\ + k_2(\alpha+\gamma)\lambda_0^2\lambda_2^2 + k_2(\beta-\delta)\lambda_1^2\lambda_3^2 + b\lambda_0\lambda_1\lambda_2\lambda_3 + \\ + k_3(\alpha+\beta)\lambda_0^2\lambda_3^2 \pm k_3(\delta-\gamma)\lambda_1^2\lambda_2^2 + c\lambda_0\lambda_1\lambda_2\lambda_3, \tag{5.13}$$

where the quantities a, b, c satisfy the condition:

$$a+b+c = k_1(\gamma-\beta-\alpha-\delta) + \\ + k_2(\alpha+\gamma+\beta-\delta) + k_3(\delta-\gamma\pm\alpha\pm\beta). \tag{5.14}$$

We require that each of the three quadratic forms in relation (5.13) be fixed in sign; the following conditions must then be satisfied:

$$\left.\begin{aligned} a^2 &\leqslant 4k_1^2(\alpha+\delta)(\beta-\gamma) = A^2 > 0, \\ b^2 &\leqslant 4k_2^2(\alpha+\gamma)(\beta-\delta) = B^2 > 0, \\ c^2 &\leqslant \pm 4k_3^2(\alpha+\beta)(\delta-\gamma) = C^2 > 0. \end{aligned}\right\} \tag{5.15}$$

In accordance with the necessary conditions (5.12), A, B and C are real quantities. The conditions (5.15) can also be written in the form: /202

$$\left.\begin{aligned} -A &\leqslant a \leqslant A, \\ -B &\leqslant b \leqslant B, \\ -C &\leqslant c \leqslant C. \end{aligned}\right\} \tag{5.16}$$

The obtained inequalities are satisfied for any a, b and c lying inside the parallelapiped with sides A, B, C. If the additional relation (5.14) is taken into account, then the inequalities (5.16) can be satisfied if and only if the plane d = a + b + c intersects the parallelapiped ABC, i.e., when:

$$-(A+B+C) \leqslant d \leqslant (A+B+C).$$

The last inequality is obtained by adding the inequalities (5.16); by replacing the inequalities A,B,C, and a,b,c in accordance with (5.14) and (5.15), we obtain:

$$\begin{aligned}
&-2k_1\sqrt{(\alpha+\delta)(\beta-\gamma)}-2k_2\sqrt{(\alpha+\gamma)(\beta-\delta)}-\\
&\quad-2k_3\sqrt{\pm(\alpha+\beta)(\delta-\gamma)}\leqslant k_1(\gamma-\beta-\alpha-\delta)+\\
&\quad+k_2(\alpha+\gamma+\beta-\delta)+k_3(\delta-\gamma\pm\alpha\pm\beta)\leqslant\\
&\leqslant 2k_1\sqrt{(\alpha+\delta)(\beta-\gamma)}+2k_2\sqrt{(\alpha+\gamma)(\beta-\delta)}+\\
&\qquad+2k_3\sqrt{\pm(\alpha+\beta)(\delta-\gamma)}.
\end{aligned}$$

Let us investigate the obtained expression; it can be divided into the following two inequalities:

$$\left.\begin{aligned}
&-k_1\left[(\alpha+\delta)+(\beta-\gamma)-2\sqrt{(\alpha+\delta)(\beta-\gamma)}\right]+\\
&\quad+k_2\left[(\alpha+\gamma)+(\beta-\delta)+2\sqrt{(\alpha+\gamma)(b-\delta)}\right]\pm\\
&\pm k_3\left[(\alpha+\beta)\pm(\delta-\gamma)\pm2\sqrt{\pm(\alpha+\beta)(\delta-\gamma)}\right]\geqslant 0,\\
&k_1\left[(\alpha+\delta)+(\beta-\gamma)+2\sqrt{(\alpha+\delta)(\beta-\gamma)}\right]-\\
&\quad-k_2\left[(\alpha+\gamma)+(\beta-\delta)-2\sqrt{(\alpha+\gamma)(\beta-\delta)}\right]\mp\\
&\mp k_3\left[(\alpha+\beta)\pm(\delta-\gamma)\mp2\sqrt{\pm(\alpha+\beta)(\delta-\gamma)}\right]\geqslant 0.
\end{aligned}\right| \qquad (5.17)$$

We set $\alpha+\delta=\beta-\gamma$; then $\alpha+\gamma=\beta-\delta$; this equality does not contradict the necessary conditions for stability (5.12) considered above. Then the first inequality (5.17) takes the form: /203

$$k_2\geqslant\mp k_3\frac{\left[\sqrt{\alpha+\beta}\pm\sqrt{\pm(\delta-\gamma)}\right]^2}{4(\alpha+\gamma)}, \qquad (5.18)$$

and the second inequality (5.17) reduces to the following:

$$k_1\geqslant\pm k_3\frac{\left[\sqrt{\alpha+\beta}\mp\sqrt{\pm(\delta-\gamma)}\right]^2}{4(\alpha+\delta)}. \qquad (5.19)$$

We consider the case of the upper sign, which corresponds to specifying $\Omega_{\kappa 3}$ by the third equality of (5.9). The inequality (5.18) is always satisfied for any positive k_2 and k_3. We impose

another limitation on the parameters $\alpha, \beta, \gamma, \delta$ in the inequality (5.19): we set $\gamma = \alpha + \delta - \beta$. Then this inequality takes the form:

$$k_1 \geqslant k_3 \frac{\beta - \sqrt{\beta^2 - \alpha^2}}{2(\alpha + \delta)}, \quad \beta \geqslant \alpha,$$

and all the necessary conditions (5.12) are satisfied. This inequality can always be satisfied by the selection of the remaining unfixed parameter α.

By analogy, for the case of lower sign, which corresponds to specifying $\Omega_{к3}$ by the fourth equality of (5.9), the inequality (5.19) is satisfied identically. By setting $\gamma = \alpha + \delta - \beta$, we reduce the inequality (5.18) to the form:

$$k_2 \geqslant k_3 \frac{\alpha - \sqrt{\alpha^2 - \beta^2}}{2(\alpha + \gamma)}, \quad \alpha \geqslant \beta.$$

Obviously, this inequality also can always be satisfied by the selection of the parameters $\alpha \geqslant \beta \geqslant \gamma$. Hence, it follows that the inequalities (5.17) can be satisfied with the selection of the parameters α, β, δ and γ for any coefficients k_i, which corresponds to fixing ther sign of the derivative (5.11) and, consequently, to the asymptotic stability of control by the direction cosines.

Nowhere in the presented proof has the magniture of the coefficients k_i ($i = 1,2,3$) been fixed or the property of their constancy been used. In accordance with the principle of "apparent linearization", any nonlinear control function satisfying the conditions (5.8) with $x \neq 0$ can be represented as: / 204

$$F_i(x) = k_i(x)\, x \qquad (i = 1, 2, 3).$$

Hence, it follows that any nonlinear function of the direction cosines of the type (5.8) will also ensure stable orientation.

By analogy, one can consider the case of control when the direction cosines of the other two axes $\mathbf{i}_2$ and $\mathbf{i}_3$ or $\mathbf{i}_1$ and $\mathbf{i}_3$ are used for correction.

III. Three independent correction signals, which are specified as a function of the direction cosines or the quantities $\lambda_0\lambda_i$, are sufficient for stable control of the rigid body position. Let us establish what minimum number of independent signals is necessary for stable orientation, i.e., in other words, whether three independent control signals are excessive. To answer this question, we will investigate what position occurs for the coordinate system E with the use of one or two correction signals.

We will consider, as an example, control by the quaternion components (5.6) and assume that only one correction signal is used $k_1 > 0$, but $k_2 = k_3 = 0$ (i.e., control along one channel). In this case, the derivative of the Lyapunov function (5.5) equals $k_1\lambda_0^2\lambda_1^2$ and vanishes everywhere that $\lambda_1 = 0$, except the position $\lambda_0 = 0$. Obviously, these positions will be equilibrium positions in which the control (the correction angular velocity) vanishes. This case corresponds to the position of the system **E**, when the finite rotation vector carrying the basis I into the basis **E** lies in the plane e_2e_3; any of these positions corresponds to the entire trajectory of the equations of motion (5.3).

By analogy, with the use of two correction signals, for example, such that $k_1 > 0$, $k_2 > 0$ and $k_3 = 0$ (control along two channels), we have the derivative of (5.5) equaling zero for $\lambda_1 = \lambda_2 = 0$ which occurs when the finite rotation vector is directed along the e_3 axis. Any of these positions is also an equilibrium position. Hence, it follows that to orient a body relative to an inertial coordinate system, it is necessary to / 205 use no less than three independent correction signals, i.e., three

independent angular position sensor signals.

§ 5.3. KINEMATIC PROBLEM OF ORIENTATION IN A ROTATING COORDINATE SYSTEM

Besides the considered case of the inertial orientation, one most frequently encounters orientation in a rotating coordinate system. An example of such is the orientation of spacecraft in the so-called orbital coordinate system, the directions of whose axes are linked with the orbital plane and the direction toward the center of the planet. The orbital coordinate system performs an orderly rotation with the angular velocity ω_0 equaling the orbital angular velocity. The vector ω_0 has a definite direction in the orbital system axes: it is constant in direction and is placed along the binormal to the orbit. For such motion, the orbital coordinate system performs a rotation in one plane relative to inertial space (precession of the orbital plane is neglected).

Orientation of spacecraft toward another craft can serve as another example of orientation in a rotating coordinate system. If one considers the relative motion of two crafts, then the radius vector joining them performs (in the general case) a rotation in one moving plane. Accordingly, one can introduce a coordinate system connected with the plane of rotation, one axis of which is positioned along the radius vector, the other is perpendicular to the plane of rotation. In this case, the rotational angular velocity of the reference basis relative to inertial space has a fixed position in the axes of the reference basis.

I. Thus, for consideration of the problem of orientation in a rotating coordinate system, we will assume that the body coordinate system E coincides with the reference system I by imparting

the correction angular velocity to the basis E. The coordinate system I then rotates relative to inertial space with the angular / 206 velocity ω_0. The angular velocity vector is then in a fixed position in the basis J and does not change its direction. Without decreasing the generality, we will assume that the vector $\omega_0 = \omega_0 i_3$ is directed along the i_3 axis.

We will consider what absolute angular velocity the coordinate system E must have. Firstly, the rotational velocity of the basis E must obviously contain the correction angular velocity $\Omega_к$ specified as the function of the mismatch of the bases E and I with the help of the projections $\Omega_{кj}$ in the body axes. Secondly, it is advisable to require that the oriented position be the equilibrium position. For this, it is necessary that with the coincidence of the coordinate systems E and I, when the correction angular velocity vanishes, the basis E performs the same rotation as the reference basis. Thus, it is necessary that the basis E has the additional rotational velocity ω^* directed along the same axis (e_3) as the reference basis (i_3) and having the same magnitude, i.e., one must have $\omega^* = \omega^* e_3 = \omega_0 e_3$. Hence, it follows that the absolute angular velocity of the basis E equals:

$$\Omega = \Omega_к + \omega^* = \Omega_{к1} e_1 + \Omega_{к2} e_2 + (\Omega_{к3} + \omega_0) e_3. \qquad (5.20)$$

Let the position of the body basis E relative to the reference be specified by the quaternion $\Lambda(t)$. In addition, we will introduce the inertial basis Z and describe the motion of the bases E and I relative to it with the help of the quaternions N and M, respectively. If the transition from the basis Z is given by the quaternion M, and from the basis I to the basis E by the quaternion Λ, then the transition from the basis Z to the basis E is determined by their product:

$$\mathbf{N} = \mathbf{M} \circ \Lambda. \qquad (5.21)$$

The kinematic equations, which determine the motion of the bases I and E relative to the inertial basis, will have the following form, in accordance with (3.11), (3.12) and with consideration of (5.20):

$$\left.\begin{aligned} 2\dot{\mathbf{M}} &= \omega_{0Z} \circ \mathbf{M} = \mathbf{M} \circ \omega_{0I}, \\ 2\dot{\mathbf{N}} &= \Omega_Z \circ \mathbf{N} = \mathbf{N} \circ \Omega_E, \end{aligned}\right\} \tag{5.22}$$

where ω_{0Z}, ω_{0I} are the hypercomplex mappings of the vector ω_0 onto the bases Z and I, and Ω_Z and Ω_E are the mappings of the vector Ω (5.20) onto the bases Z and E.

In accordance with (5.21) / 207

$$\Lambda = \tilde{\mathbf{M}} \circ \mathbf{N},$$

from which one can find the kinematic equation determining the motion of the basis E relative to the basis I:

$$\begin{aligned} 2\dot{\Lambda} &= 2\tilde{\mathbf{M}} \circ \dot{\mathbf{N}} - 2\tilde{\mathbf{M}} \circ \dot{\mathbf{M}} \circ \tilde{\mathbf{M}} \circ \mathbf{N} = \\ &= \tilde{\mathbf{M}} \circ \mathbf{N} \circ \Omega_E - \tilde{\mathbf{M}} \circ \mathbf{M} \circ \omega_{0I} \circ \tilde{\mathbf{M}} \circ \mathbf{N} = \Lambda \circ \Omega_E - \omega_{0I} \circ \Lambda. \end{aligned} \tag{5.23}$$

We note that this is the most convenient form of the equation, since the vector Ω is specified by its components in the basis E, i.e., the vector Ω_E is actually given. In exactly the same way, the vector ω_0 is given in the basis I, i.e., the quantity ω_{0I} is known.

If, for example, the vector ω_0 is reprojected onto the basis E, then we obtain:

$$\omega_{0E} = \tilde{\Lambda} \circ \omega_{0I} \cdot \Lambda,$$

and Equation (5.23) takes the usual form (3.11) for the angular velocity $\Omega_E - \omega_{0E}$, which is the relative velocity of motion of the bases E and I. In accordance with (5.20) and by definition of the vector ω_0, we have the following magnitudes of the mappings:

$$\Omega_E = (\Omega_{K1},\ \Omega_{K2},\ \Omega_{K3} + \omega_0), \qquad \omega_{0I} = (0,\ 0,\ \omega_0).$$

With consideration of this kinematic Equation (5.23) of motion in quaternion components, Λ is written in the form of the scalar relations:

$$\left.\begin{aligned} 2\dot{\lambda}_0 &= -\lambda_1\Omega_{K1} - \lambda_2\Omega_{K2} - \lambda_3\Omega_{K3}, \\ 2\dot{\lambda}_1 &= \lambda_0\Omega_{K1} + \lambda_2\Omega_{K3} - \lambda_3\Omega_{K2} + 2\lambda_2\omega_0, \\ 2\dot{\lambda}_2 &= \lambda_0\Omega_{K2} + \lambda_3\Omega_{K1} - \lambda_1\Omega_{K3} - 2\lambda_1\omega_0, \\ 2\dot{\lambda}_3 &= \lambda_0\Omega_{K3} + \lambda_1\Omega_{K2} - \lambda_2\Omega_{K1}. \end{aligned}\right\} \qquad (5.24)$$

The obtained equations for orientation in a rotating coordinate system differs from the analogous equations for inertial orientations (5.3) by the presence of two terms with angular velocity ω_0. As is seen, Equations (5.24) fall into two groups: equations for the components λ_0 and λ_3, which are analogous to the same Equations (5.3), and equations for λ_1 and λ_2, which are linked because of the presence of the rotational velocity of the reference basis ω_0. Obviously, the equilibrium position $\lambda_i = 0$, $\Omega_{Ki} = 0$ $(i = 1, 2, 3)$ is a solution of the equation, which would be obtained when ω^* is set equal to ω_0. One can verify that in the opposite case, terms with $\omega_0 - \omega^*$ would appear in the first and fourth equations of (5.24). / 208

It is interesting to note that, for the constant quantities $\omega_0 = \omega^*$ and $\Omega_K = 0$ (5.23) represents a solveable [cf., (3.18)] case of precession:

$$\lambda_0(t) = \lambda_0^0, \quad \lambda_3(t) = \lambda_3^0,$$
$$\lambda_1(t) = \lambda_1^0 \cos\omega_0 t + \lambda_2^0 \sin\omega_0 t,$$
$$\lambda_2(t) = \lambda_2^0 \cos\omega_0 t + \lambda_1^0 \sin\omega_0 t.$$

This solution indicates that, without correction, the motion of the coordinate system E relative to the system I is such that the finite rotation vector (giving the transition from the system I to the system E) rotates on a cone, whose axis lies along the

axis e_3 if the motion is observed in the body axes, or along the axis i_3 if observed from the reference basis.

II. We will investigate the necessary and sufficient conditions for the stability of the process of reducing the coordinate system E to the system I. We will consider the case when the control is accomplished by the quaternion components. We introduce, as in the preceding paragraph, the Lyapunov function in the form (5.4). Its derivative composed in accordance with Equations (5.24) have exactly the same form as the case of inertial orientation (5.5). Hence, it follows at once that the linear control (5.6) and the arbitrary nonlinear control of the form (5.7) satisfying the conditions (5.8) accomplish asymptotic convergence of the correction process.

In some cases, in contrast to the inertial orientation, rotation of the reference coordinate system makes it possible to reduce the coordinate system E to the system I with control by only two signals. Such a possibility for satisfying the orientation with incomplete information from angular position sensors is caused by the change of the equilibrium position with rotation. / 209
This fact has been indicated in a number of works [42, 45] without investigating the possible types of corrections and without proving the stability of the reduction process as a whole.

We will consider the case of correction by two signals Ω_{k1} and Ω_{k2} ($k_1 > 0, k_2 > 0, k_3 = 0$) . In this case, the derivative of the Lyapunov function is negative everywhere, except the region $\lambda_1 = \lambda_2 = 0$. It is necessary to determine whether the set $\lambda_1 \equiv \lambda_2 \equiv 0$ contains entire trajectories satisfying Equations (5.24) for the given correction. By setting the quantity $\Omega_{k1} = 0$, in Equations (5.24), we obtain:

$$\dot{\lambda}_0 = 0, \quad \dot{\lambda}_3 = 0, \quad \dot{\lambda}_1 = \lambda_2\omega_0, \quad \dot{\lambda}_2 = -\lambda_1\omega_0. \tag{5.25}$$

Obviously, for the case $\lambda_1 \equiv \lambda_2 \equiv 0$, the trajectories $\lambda_0(t) = \lambda_0^0$, $\lambda_3(t) = \lambda_3^0$ satisfy these equations. Hence, it follows that, with correction by the two signals Ω_{K1} and Ω_{K2} [the type (5.6) or (5.7)], there occur entire trajectories on which the derivative of the Lyapunov function vanishes, i.e., the conditions of asymptotic stability are not satisfied, and consequently, it is impossible to accomplish orientation with the use of these two signals.

Now let the correction be carried out by the following pair of signals by Ω_{K1} and Ω_{K3} ($k_1 > 0$, $k_3 > 0$, $k_2 = 0$). We will investigate whether for such correction there can exist solutions for which Ω_{K1} and Ω_{K3} vanish when any other coordinates differ from zero. By setting $\Omega_{Ki} \equiv 0$, we obtain the relations (5.25), which obviously can satisfy the conditions $\lambda_1 \equiv \lambda_3 \equiv 0$ only when $\lambda_2 = 0$. Thus, the set $\lambda_1 \equiv \lambda_3 \equiv 0$ does not contain entire trajectories, except the oriented position $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence, it follows that, for correction by these two signals, it is possible to accomplish spatial orientation.

By analogy, entire trajectories, except for the point $\lambda_1 = \lambda_2 = \lambda_3 = 0$, are not contained in the set $\lambda_2 \equiv \lambda_3 \equiv 0$ when correcting by the signals Ω_{K2} and Ω_{K3} ($k_2 > 0$, $k_3 > 0$, $k_1 = 0$), i.e., it is also possible to accomplish orientation with this pair of signals.

Thus, orientation in a rotating coordinate system can be accomplished by two signals of angular position sensors. Then control of the type (5.6) or (5.7) ensures stability as a whole / 210
(as previously, this control must be redefined at the point such that it differs from zero).

When correcting by only one signal, there exists whole trajectories on which the derivative $\dot{W}$ equals zero. Thus, when correcting by $\Omega_{к1}$ $(k_2 = k_3 = 0,\ k_1 > 0)$, we have the following solution for the set $\lambda_1 \equiv 0$ (on which $\Omega_{к1}$ vanishes):

$$\lambda_0(t) = \lambda_0^0, \quad \lambda_3(t) = \lambda_3^0.$$

Analogous solutions also exist for correction by the signal $\Omega_{к2}$, i.e., for $k_1 = k_3 = 0,\ k_2 > 0$. For the case of correction by the signal $\Omega_{к3}$, i.e., for $k_1 = k_2 = 0,\ k_3 > 0$, we have the t $\lambda_3 \equiv 0$ the trajectories:

$$\lambda_1(t) = \lambda_1^0 \cos \omega_0 t + \lambda_2^0 \sin \omega_0 t,$$
$$\lambda_2(t) = \lambda_2^0 \sin \omega_0 t - \lambda_1^0 \sin \omega_0 t,$$

which satisfy Equations (5.25). Hence, it follows that it is impossible to accomplish stable orientation with the use of one angular position sensory signal.

III. It is interesting to illustrate the case considered above of stable correcting by the signals $\Omega_{к3}$ and $\Omega_{к1}$ (i.e., according to $\lambda_0\lambda_3$ and $\lambda_0\lambda_1$) by investigating the reduction process for the case of linear correction (5.6). We note at once that the correction process also takes place by the signals $\Omega_{к3}$ and $\Omega_{к2}$ $(\lambda_0\lambda_3,\ \lambda_0\lambda_2)$ in an analogous manner.

Let the correction coefficients k_i be equal and the orientation be performed with the control:

$$\Omega_{к1} = -k\lambda_0\lambda_1, \quad \Omega_{к3} = -k\lambda_0\lambda_3.$$

In this case, Equations (5.24) take the following form:

$$\left.\begin{aligned} 2\dot{\lambda}_0 &= k\lambda_0(\lambda_1^2 + \lambda_3^2) = k\lambda_0(1 - \lambda_0^2 - \lambda_2^2), \\ 2\dot{\lambda}_3 &= -k\lambda_0(\lambda_0\lambda_3 - \lambda_1\lambda_2), \\ 2\dot{\lambda}_1 &= -k\lambda_0(\lambda_0\lambda_1 + \lambda_2\lambda_3) + 2\omega_0\lambda_2, \\ 2\dot{\lambda}_2 &= -2\omega_0\lambda_1. \end{aligned}\right\} \qquad (5.26)$$

These equations describe the process of reducing the basis **E** to the basis **I** with correction by two signals; the exact solution of these equations cannot be obtained. In connection with this, we / 211 will present an approximation investigation of the equations (5.26). The motion described by (5.26) can be divided into three parts determined by the initial conditions and, consequently, by the region of variation of the quantities λ_i.

1. Let the initial mismatch and correction coefficient k be such that the condition $\Omega_{\kappa i} \gg \omega_0$ is satisfied everywhere, except at those positions where λ_1 and λ_3 are small quantities. (This condition is equivalent to the requirement $\omega_0 \ll k$.) We will consider the following motion close to (5.26) and differing from it by the small terms with ω_0, which can be neglected according to the assumption made:

$$\left.\begin{aligned} 2\dot{\lambda}_0 &= k\lambda_0(1-\lambda_0^2-\lambda_2^2),\\ 2\dot{\lambda}_3 &= -k\lambda_0(\lambda_0\lambda_3-\lambda_1\lambda_2),\\ 2\dot{\lambda}_1 &= -k\lambda_0(\lambda_1\lambda_0+\lambda_2\lambda_3),\\ 2\dot{\lambda}_2 &= 0. \end{aligned}\right\} \qquad (5.27)$$

These equations can be integrated; with consideration of the fact that $\lambda_2 = \lambda_2^0$ (λ_i^0 are the initial values of the quaternion components), we have from the first Equation of (5.27):

$$\lambda_0^2(t) = \frac{(\lambda_0^0)^2\left[1-(\lambda_2^0)^2\right]}{(\lambda_0^0)^2\left\{1-e^{-k\left[1-(\lambda_2^0)^2\right]t}\right\}+\left[1-(\lambda_2^0)^2\right]e^{-k\left[1-(\lambda_2^0)^2\right]t}}.$$

The obtained solution is sufficient to analyze the nature of the motion. Since $\lambda_1^2+\lambda_3^2 = 1-\lambda_2^2-\lambda_0^2$, then:

$$\begin{aligned}\lambda_1^2(t)+\lambda_3^2(t) &= \left[(\lambda_1^0)^2+(\lambda_3^0)^2\right]\times\\ &\times\frac{\left[1-(\lambda_2^0)^2\right]e^{-k\left[1-(\lambda_2^0)^2\right]t}}{\left[1-(\lambda_2^0)^2\right]e^{-k\left[1-(\lambda_2^0)^2\right]t}+(\lambda_0^0)^2\left\{1-e^{-k\left[1-(\lambda_2^0)^2\right]t}\right\}} =\\ &= \left[(\lambda_1^0)^2+(\lambda_3^0)^2\right]\left\{1+\frac{(\lambda_0^0)^2}{1-(\lambda_2^0)^2}\left[e^{k\left[1-(\lambda_2^0)^2\right]t}-1\right]\right\}^{-1}.\end{aligned}$$

It follows from this equality that, for large values of t: / 212

$$\frac{\lambda_1^2(t)+\lambda_3^2(t)}{(\lambda_1^0)^2+(\lambda_3^0)^2} \approx \frac{1-(\lambda_2^0)^2}{(\lambda_2^0)^2} e^{-k\left[1-(\lambda_2^0)^2\right]t}. \qquad (5.28)$$

i.e., the convergence of the process is rather fast and is determined by the selection of the correction coefficient k and the initial conditions $\lambda_2^0 < 1$.

Obviously, relation (5.28) describes the process of reducing the axis $\mathbf{e}_2$ to the axis $\mathbf{i}_2$. We note that the closeness of λ_2^0 to one indicates that λ_0^0 is small and the position of the body is close to point $\lambda_0^0 = 0$; this case should be examined separately.

If the quaternion components are represented in the form:

$$\lambda_0 = \cos\frac{\Phi}{2}, \quad \lambda_i = \gamma_i \sin\frac{\Phi}{2} \qquad (i = 1, 2, 3),$$

where γ_i are the direction cosines of the finite rotation vector, then it follows from the relation $\lambda_1^2 + \lambda_3^2 \to 0$, that $\gamma_1^2 + \gamma_3^2 \to 0$ and, consequently, $\gamma_2^2 \to 1$. This indicates that, in the process of reducing the axis $\mathbf{e}_2$ to the axis $\mathbf{i}_2$, the rotation angle $\Phi(t)$ (the angle of Euler plane rotation) decreases from the initial value Φ_0 such that:

$$\sin\frac{\Phi(t)}{2} \to \sin\frac{\Phi_0}{2}\gamma_2^0,$$

where γ_2^0 is the initial value of the corresponding cosine of the finite rotation vector. For this situation, the coordinate system E appears to be rotated from the oriented position about the axis $\mathbf{e}_2$ by the angle $\Phi(t)$. The first, relatively rapid stage of reduction, which includes the superposition of the axis $\mathbf{e}_2$ with the axis $\mathbf{i}_2$, concludes with this.

2. The axes $\mathbf{e}_2$ and $\mathbf{i}_2$ remain close with the subsequent motion, i.e., the motion will be a rotation of the coordinate system E about the axis $\mathbf{i}_2$, as a result of which superposition of the systems E and I must be produced, i.e., the quantity $\theta(t)$ must decrease to zero. For the approximation study of the second stage of the motion, which is a rotation about the axis $\mathbf{e}_2(\mathbf{i}_2)$, / 213 we set the derivatives $\dot{\lambda}_1=\dot{\lambda}_3=0$ in Equations (5.27), since the quantities $\lambda_1(t)$ and $\lambda_3(t)$ are reduced to small quantities as a result of the correction, as which they remain throughout the subsequent motion. It follows from this condition that the approximate equalities can be satisfied:

$$\left.\begin{aligned} &\lambda_3 \approx \frac{\lambda_1\lambda_2}{\lambda_0}, \quad \lambda_1\lambda_0+\lambda_3\lambda_2 \approx \frac{2\omega_0\lambda_2}{k\lambda_0}, \\ &\lambda_1=\frac{2\omega_0\lambda_2}{k\left(\lambda_0^2+\lambda_2^2\right)} \approx \frac{2\omega_0}{k}\lambda_2, \end{aligned}\right\} \tag{5.29}$$

since λ_2 and λ_3 are small quantities whose squares can be neglected. By substituting the obtained value λ_1 into the fourth equation of (5.27), we have:

$$\dot{\lambda}_2=-\frac{2\omega_0^2}{k}\lambda_2,$$

from which follows the solution:

$$\lambda_2(t)=\lambda_2^0 e^{-\frac{2\omega_0^2}{k}t}, \tag{5.30}$$

which describes a turn of the basis E about the axis $\mathbf{i}_2$ toward the oriented position (here λ_2^0 is the initial value at the second stage of motion, i.e., the final value for the first stage equaling $\gamma_2^0\sin\frac{\theta}{2}$).

It is interesting to note that the magnitude of the correction coefficient k appears to have an inverse effect on the rate of convergence of the process here: the rate decreases in proportion to the increase of the coefficient k (in comparison with the first stage of motion, where the rate of axis reduction increases with increasing coefficient k).

We propose the following method for accelerating the correction: since the coordinates λ_1 and λ_2 appear "coupled," because of the presence of the velocity ω_0, and correction in this case is accomplished by the signal λ_1 ($\Omega_{K1} = k\lambda_0\dot{\lambda}_1$), then one can introduce the so-called cross correction in the form:

$$\Omega_{K2} = -h\lambda_0\lambda_1. \tag{5.31}$$

Putting aside the question of investigating the convergence of the correction process at the stage of superposition of the axis e_2 with the axis i_2 (such superposition is accomplished), we will consider only rotation about the axis e_2, i.e., motion along the coordinate λ_2. From the second and third equations of (5.24) with the selected correction for $\dot{\lambda}_1 = \dot{\lambda}_3 = 0$, we have, with consideration of first order terms in the quantities λ_1 and λ_3, a relation for the component λ_1 in a form analogous to (5.29). Then the fourth equation of (5.24) takes the following form: / 214

$$2\dot{\lambda}_2 = -(h\lambda_0^2 + 2\omega_0)\lambda_1 = -\frac{2h(\lambda_0^2 + 2\omega_0)\omega_0}{k}\lambda_2.$$

By keeping in mind that $\lambda_0^2 \approx 1 - \lambda_2^2$, we obtain the following solution of this equation:

$$\lambda_2(t) = \lambda_2^0 \frac{e^{-\frac{(h+2\omega_0)\omega_0}{k}t}}{\sqrt{1 - \frac{h}{h+2\omega_0}(\lambda_2^0)^2\left[e^{-\frac{2\omega_0(h+2\omega_0)}{k}t} - 1\right]}}. \tag{5.32}$$

Hence, it is seen that the convergence of the process can be significantly accelerated by the introduction of cross correction.

3. After the component $\lambda_2(t)$ becomes small, the approximate solutions (5.28), (5.30), and (5.32) lose their validity. The third and the last stage of motion close to the equilibrium position for small values of λ_2 and λ_3 is described by the

linearized Equations (5.26), in which terms of second order relative to $\lambda_1, \lambda_2, \lambda_3$ ($\lambda_0 \approx 1$) are not taken into account:

$$\left.\begin{aligned} 2\dot{\lambda}_3 &= -k\lambda_3, \\ 2\dot{\lambda}_1 &= -k\lambda_1 + 2\omega_0\lambda_2, \\ 2\dot{\lambda}_2 &= -2\omega_0\lambda_1. \end{aligned}\right\} \qquad (5.33)$$

These equations can be used for analyzing the accuracy of the reduction process. Equations (5.33) indicate that the channel λ_3 can be independent of the channels λ_1 and λ_2, which are interconnected because of the presence of the rotation velocity ω_0. The characteristic equation for the second and third equations of / 215 the system (5.33) will be:

$$p^2 + \frac{k}{2}p + \omega_0^2 = 0$$

and has the roots:

$$p_{1,2} = -\frac{k}{4} \pm \sqrt{\frac{k^2}{16} - \omega_0^2}.$$

For very small magnitudes of ω_0 (compared to k), the length of the convergence process is large. To accelerate the process, one can introduce the cross correction (5.31); in this case, the third equation of (5.33) is changed to the following:

$$2\dot{\lambda}_2 = -h\lambda_1 - 2\omega_0\lambda_1.$$

The characteristic equation takes the form:

$$p^2 + \frac{k}{2}p + \omega_0\left(\omega_0 + \frac{h}{2}\right) = 0$$

and will have the roots:

$$p_{1,2} = -\frac{k}{4} \pm \sqrt{\frac{k^2}{16} - \omega_0\left(\omega_0 + \frac{h}{2}\right)}.$$

Hence, it is seen that the rate of convergence can be increased by the choice of the cross correction coefficient h.

In conclusion, we will consider the behavior of the system at the singluar point $\lambda_2^0 \approx 1$, at which, as follows from the solution (5.28), there is no convergence of the process. The value $\lambda_2^0 = 1$ corresponds to the position when the axes $\mathbf{e}_2$ and $\mathbf{i}_2$ of the coordinate systems E and I coincide, and the axes $\mathbf{e}_1$ and $\mathbf{e}_3$ coincide with the axes $-\mathbf{i}_1$ and $-\mathbf{i}_3$, i.e., the coordinate systems E and I are turned about by 180°. We note that the exact equations of motion (5.26) allow the particular solution $\lambda_2 = 1$, $\lambda_0 = \lambda_1 = \lambda_3 = 0$, and this position is the equilibrium position. The correcting signals $\Omega_{к1}$ and $\Omega_{к3}$ vanish at this position. If the equation is redefined at this point so that it does not disappear, then the position will no longer be an equilibrium position according to the Equations (5.24).

To investigate the vicinity of this point, we set $\lambda_2 \approx 1$ in Equations (5.24) and we will consider the quantities λ_0, λ_1 and λ_3 to be small. In this case, we obtain the following system of equations (recall that $\Omega_{к2} \equiv 0$): / 216

$$
\begin{aligned}
2\dot{\lambda}_0 &= -\lambda_1\Omega_{к1} - \lambda_3\Omega_{к3},\\
2\dot{\lambda}_1 &= \lambda_0\Omega_{к1} + \Omega_{к3} + 2\omega_0,\\
2\dot{\lambda}_2 &= \lambda_3\Omega_{к1} - \lambda_1\Omega_{к3} - 2\lambda_1\omega_0,\\
2\dot{\lambda}_3 &= \lambda_0\Omega_{к3} - \Omega_{к1}.
\end{aligned}
$$

It is seen from these equations that, in spite of the smallness of the quantities λ_0, λ_1, λ_3, the rate of motion along the coordinates λ_3 and λ_1 is determined mainly by the correction velocities $\Omega_{к1}$ and $\Omega_{к3}$, respectively. When the correction velocities are given in the form (5.6) and vanish at the point $\lambda_0 = \lambda_1 = \lambda_3 = 0$, motion in a small neighborhood of this point is described by the approximate equations:

$$\left.\begin{aligned} 2\dot{\lambda}_0 &= 0, \\ 2\dot{\lambda}_3 &= k\lambda_0\lambda_1, \\ 2\dot{\lambda}_1 &= -k\lambda_0\lambda_3 + 2\omega_0, \\ 2\dot{\lambda}_2 &= -2\omega_0\lambda_1, \end{aligned}\right\} \qquad (5.34)$$

obtained from (5.26) with consideration of terms to the third order relative to the quantities λ_0, λ_1, λ_3.

One can see that the Equations (5.34) describe the motion of a conservative system, i.e., the quantities λ_i vary according to a harmonic law with the frequency

$$\nu^2 = \frac{k^2(\lambda_0^0)^2}{4},$$

where λ_0^0 is the initial value of the coordinate λ_0. However, the quantities λ_i do not remain small in the process of motion for all cases. Since the drift of the trajectory from the point $\lambda_2 = 1$ is characterized, to a large degree, by the change of the coordinate $\lambda_2(t)$, we present the solution of the systems (5.34) for this coordinate

$$\lambda_2(t) = \lambda_2^0 \frac{\frac{k^2(\lambda_0^0)^2}{4} + \omega_0^2 \cos \nu t}{\nu^2} + \\ + \lambda_3^0 \frac{\omega_0 k \lambda_0^0}{2\nu^2}(1 - \cos \nu t) - \lambda_1^0 \frac{\omega_0}{\nu} \sin \nu t.$$

This solution shows that the coordinate $\lambda_2(t)$ remains to close to one, only for / 217

$$\frac{k^2(\lambda_0^0)^2}{4} \gg \omega_0^2$$

Even with small coefficients, the presence of the rotational angular velocity ω_0 leads to a comparatively rapid drift of the system from this point.

IV. We consider the question of control stability, when the controlling correction signals are given as a function of the direction cosines. We find the Lyapunov function in the form (5.10) for the control, which is accomplished by the direction cosines of the axes $\mathbf{i}_1$ and $\mathbf{i}_2$ and has the form (5.9). The derivative of the Lyapunov function composed in accordance with the equations of motion (5.24) will equal, for this control:

$$\begin{aligned}-\tfrac{1}{2}\dot{W} = {} & k_1(\alpha+\delta)\lambda_0^2\lambda_1^2 + k_1(\beta-\gamma)\lambda_2^2\lambda_3^2 + k_2(\alpha+\gamma)\lambda_0^2\lambda_2^2 + \\ & + k_2(\beta-\delta)\lambda_1^2\lambda_3^2 + k_3(\alpha+\beta)\lambda_0^2\lambda_3^2 \pm k_3(\delta-\gamma)\lambda_1^2\lambda_2^2 + \\ & + k_1[(\gamma-\alpha-\beta-\delta) + k_2(\alpha+\beta+\gamma-\delta) \pm \\ & \pm k_3(\delta-\gamma+\alpha+\beta)]\lambda_0\lambda_1\lambda_2\lambda_3 - 2\omega_0(\delta-\gamma)\lambda_1\lambda_2.\end{aligned} \tag{5.35}$$

We set $\delta = \gamma$ and require a fixed sign for the obtained quadratic form. For this, it is necessary, above all, that the conditions:

$$k_i \geqslant 0 \quad (i = 1, 2, 3), \quad \beta \geqslant \gamma \quad (\beta \geqslant \delta)$$

are satisfied. We introduce the designation:

$$d = a + b = (k_2 - k_1 \pm k_3)(\alpha+\beta) \tag{5.36}$$

and represent the derivative (5.35) in the form:

$$\begin{aligned}-\tfrac{1}{2}\dot{W} = {} & [k_1(\alpha+\delta)\lambda_0^2\lambda_1^2 + k_1(\beta-\gamma)\lambda_2^2\lambda_3^2 + a\lambda_0\lambda_1\lambda_2\lambda_3] + \\ & + [k_2(\alpha+\gamma)\lambda_0^2\lambda_2^2 + k_2(\beta-\delta)\lambda_1^2\lambda_3^2 + b\lambda_0\lambda_1\lambda_2\lambda_3] + \\ & + k_3(\alpha+\beta)\lambda_0^2\lambda_3^2.\end{aligned}$$

The conditions for a fixed sign of the two quadratic forms (conditionally enclosed in brackets) in the presented expression will be:

$$4k_1^2(\alpha+\gamma)(\beta-\gamma) \geqslant a^2, \quad 4k_2^2(\alpha+\gamma)(\beta-\gamma) \geqslant b^2.$$

The same conditions can be written in the form: / 218

$$\begin{aligned}-2k_1\sqrt{(\alpha+\gamma)(\beta-\gamma)} \leqslant a \leqslant 2k_1\sqrt{(\alpha+\gamma)(\beta-\gamma)}, \\ -2k_2\sqrt{(\alpha+\gamma)(\beta-\gamma)} \leqslant b \leqslant 2k_2\sqrt{(\alpha+\gamma)(\beta-\gamma)}.\end{aligned}$$

These inequalities are satisfied if and only if the straight line (5.36) in the plane of the parameters a, b intersects the rectangle with sides $k_1\sqrt{(\alpha+\gamma)(\beta-\gamma)}$ and $k_2\sqrt{(\alpha+\gamma)(\beta-\gamma)}$, which occurs only if:

$$-2(k_1+k_2)\sqrt{(\alpha+\gamma)(\beta-\gamma)}\leqslant d\leqslant \leqslant 2(k_1+k_2)\sqrt{(\alpha+\gamma)(\beta-\gamma)}.$$

The obtained expression can be divided into two inequalities:

$$\left.\begin{aligned} k_2(\sqrt{\alpha+\gamma}+\sqrt{\beta-\gamma})^2-k_1(\sqrt{\alpha+\gamma}-\sqrt{\beta-\gamma})^2\pm \\ \pm k_3(\alpha+\beta)\geqslant 0, \\ k_1(\sqrt{\alpha+\gamma}+\sqrt{\beta-\gamma})^2-k_2(\sqrt{\alpha+\gamma}-\sqrt{\beta-\gamma})^2\mp \\ \mp k_3(\alpha+\beta)\geqslant 0. \end{aligned}\right\} \qquad (5.37)$$

In order to satisfy the inequalities (5.37), we set:

$$\alpha+\gamma=\beta-\gamma.$$

We consider the case with the upper sign. The first inequality of (5.37) is always satisfied, the second leads to:

$$k_1\geqslant k_3\frac{\alpha+\beta}{4(\alpha+\gamma)}=\frac{k_3}{2}. \qquad (5.38)$$

For the lower sign, the second inequality of (5.37) is always satisfied, and the first leads to the form:

$$k_2\geqslant k_3\frac{\alpha+\beta}{4(\alpha+\gamma)}=\frac{k_3}{2}. \qquad (5.39)$$

Obviously, these inequalities can be satisfied by the choice of correction coefficients, i.e., the derivative (5.35) of the Lyapunov function will be negative-definite and the correction process will be asymptotically stable.

As seen from the presented proof, rotation of the reference / 219
coordinate system I leads to the appearance of additional conditions on the parameters α, β, γ and δ, because of which the conditions (5.38) and (5.39) arise. The case (5.38) corresponds to the correction when the two direction cosines of the axis $\mathbf{i}_2$ are used in the control. Thus, this case defines the process of the axis $\mathbf{e}_2$ being superimposed with the axis $\mathbf{i}_2$. The analogous condition (5.39) defines the process of the axis $\mathbf{e}_1$ being superimposed with $\mathbf{i}_1$. In both cases, the signal determined by the third direction cosine (5.9) is used for the third channel.

It follows at once from the presented proof that the magnitude of the correction coefficient k_2 can be set equal to zero when using the two direction cosines of the axis $\mathbf{i}_2$ in the control. Then all the fixed sign conditions are preserved for the Lyapunov function (everywhere) and for its derivative everywhere, except positions corresponding to the coincidence of the axes $\mathbf{e}_2$ and $\mathbf{i}_2$. However, as was previously shown, these positions are not entire trajectories of the equations of motion (5.24). Hence, it follows that it is possible to accomplish three-axis orientation in a rotating coordinate system with correction by the direction cosines of one axis $\mathbf{i}_2$. By analogy, consideration of the conditions (5.39) indicates that it is possible to reduce the system **E** to the reference system with correction by the direction cosines of the axis $\mathbf{i}_1$.

In order to explain the nature of the limitations on the correction coefficients (5.38) and (5.39), we consider the case when the correcting signals are nonlinear functions of the direction cosines. We will investigate the case of correction by two signals (direction cosines) of the axis $\mathbf{i}_2$ in the body axes:

$$\left.\begin{aligned}\Omega_{x1} &= -\Omega_1 \operatorname{sign}(\lambda_0\lambda_1 - \lambda_2\lambda_3),\\ \Omega_{x3} &= -\Omega_3 \operatorname{sign}(\lambda_0\lambda_3 + \lambda_1\lambda_2),\end{aligned}\right\} \qquad (5.40)$$

i.e., we will consider the case of a relay control function.

As the Lyapunov function, we will take a function defined by the direction cosine of the axis $\mathbf{i}_2$ with the axis $\mathbf{e}_2$: / 220

$$W = \tfrac{1}{2}\left[1 - (\lambda_0^2 + \lambda_2^2 - \lambda_1^2 - \lambda_3^2)\right] = \lambda_1^2 + \lambda_3^2.$$

Its derivative, in view of Equations (5.24) with the correction functions (5.40), will equal:

$$-\dot{W} = \Omega_1 |\lambda_0\lambda_1 - \lambda_2\lambda_3| + \Omega_3 |\lambda_0\lambda_3 + \lambda_1\lambda_2| + 2\omega_0\lambda_1\lambda_2.$$

We consider the region where the obtained expression vanishes or becomes negative, i.e., the region where the motion is not curtailed. Obviously, for this it is necessary that the following inequalities be satisfied:

$$\left.\begin{aligned} -e_1\lambda_1\lambda_2 &\geqslant |\lambda_0\lambda_1 - \lambda_2\lambda_3| \\ -e_3\lambda_1\lambda_2 &\geqslant |\lambda_0\lambda_3 + \lambda_1\lambda_2| \end{aligned}\right\} \text{ for } \lambda_1\lambda_2 \leqslant 0,$$

where:

$$e_1 = \frac{2\omega_0}{\Omega_1} > 0, \quad e_2 = \frac{2\omega_0}{\Omega_3} > 0.$$

By squaring the inequalities, we obtain:

$$\left.\begin{aligned} e_1^2\lambda_1^2\lambda_2^2 &\geqslant \lambda_0^2\lambda_1^2 - 2\lambda_0\lambda_1\lambda_2\lambda_3 + \lambda_2^2\lambda_3^2, \\ e_3^2\lambda_1^2\lambda_2^2 &\geqslant \lambda_0^2\lambda_3^2 + 2\lambda_0\lambda_1\lambda_2\lambda_3 + \lambda_1^2\lambda_2^2. \end{aligned}\right\} \qquad (5.41)$$

In order that there exist regions where the derivative of the Lyapunov function is positive, it is necessary to satisfy both inequalities. Conversely: if it is shown that just one inequality is not satisfied over the whole region of motion, then the derivative will be a negative-definite function. We consider the second inequality as a quadratic trinomial relative to $\lambda_0\lambda_3$

$$(\lambda_0\lambda_3)^2 + 2(\lambda_0\lambda_3)(\lambda_1\lambda_2) + (1 - e_3^2)(\lambda_1\lambda_2)^2 \leqslant 0, \qquad (5.42)$$

having the roots:

$$(\lambda_0\lambda_3)_{1,2} = (-\lambda_1\lambda_2) \pm \varepsilon_3(\lambda_1\lambda_2) = -(1 \mp \varepsilon_3)(\lambda_1\lambda_2).$$

In view of the fact that $\lambda_1\lambda_2 \leqslant 0$, the inequality (5.42) is satisfied under the condition $\varepsilon_3 < 1$ in the interval

$$(\lambda_0\lambda_3)_1 \leqslant \lambda_0\lambda_3 \leqslant (\lambda_0\lambda_3)_2$$

or, according to the quadratic equation (5.42):

$$(-\lambda_1\lambda_2)(1-\varepsilon_3) \leqslant \lambda_0\lambda_3 \leqslant (-\lambda_1\lambda_2)(1+\varepsilon_3).$$

To evaluate the quantity $\lambda_0\lambda_1\lambda_2\lambda_3$, we multiply this inequality by $(-\lambda_1\lambda_2)$ / 221

$$\lambda_1^2\lambda_2^2(1-\varepsilon_3) \leqslant -\lambda_0\lambda_1\lambda_2\lambda_3 \leqslant \lambda_1^2\lambda_2^2(1+\varepsilon_3).$$

Hence, it follows that:

$$-\lambda_0\lambda_1\lambda_2\lambda_3 = \nu\lambda_1^2\lambda_2^2, \tag{5.43}$$

where $1-\varepsilon_3 \leqslant \nu \leqslant 1+\varepsilon_3$.

By substituting the obtained value into the second inequality of (5.41), we have

$$(2\nu - \varepsilon_1^2)\lambda_1^2\lambda_2^2 + \lambda_0^2\lambda_1^2 + \lambda_2^2\lambda_3^2 \leqslant 0. \tag{5.44}$$

It is obvious that, as a consequence of the relation (5.43), the factor:

$$2\nu - \varepsilon_1^2 \geqslant 2(1-\varepsilon_3) - \varepsilon_1^2 \tag{5.45}$$

can always be made positive by the choice of quantities ε_1 and ε_3. Then the strict inequality (5.44) cannot be satisfied. The equality (5.44) is possible only in the following cases:

a) $\lambda_1 = 0,\ \lambda_2 = 0$; b) $\lambda_1 = 0,\ \lambda_3 = 0$; c) $\lambda_2 = 0,\ \lambda_0 = 0$.

For the first of these cases, the derivative of the selected Lyapunov function

$$-\dot{W} = \Omega_3 |\lambda_0 \lambda_3|$$

vanishes only when $\lambda_3 = 0$ or when $\lambda_0 = 0$. Likewise, in the second and in the third cases, the derivative of the Lyapunov function vanishes on the set of values $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0$ and on the set $\lambda_0 = 0,\ \lambda_2 = 0$ or $\lambda_0 = 0,\ \lambda_1 = 0,\ \lambda_3 = 0$.

Thus, it has been shown that the derivative of the Lyapunov function is negative everywhere, except the two positions: 1) $\lambda_1 = \lambda_2 = \lambda_3 = 0$, where this position corresponds to accomplishment of the required orientation; 2) $\lambda_0 = 0$ and one of the other components also equals zero, where this, as noted previously, is an unstable equilibrium position. We note that the latter follows from the fact that the derivative:

$$(\lambda_0^2 + \lambda_2^2)^{\cdot} = -(\lambda_1^2 + \lambda_3^2)^{\cdot} \geqslant 0,$$

i.e., it is positive-definite for the considered components of the case a). / 222

It is interesting to note the meaning of the limitation on the quantities ε_1 and ε_3: it is necessary that the conditions $\varepsilon_1 < 1$ and $\varepsilon_3 < 1$ and a somewhat stronger condition (5.45) be satisfied. Thus, the magnitudes of the correction velocities $\Omega_1,\ \Omega_3$ must be at minimum twice as large as the rotation velocity ω_0, which, as is easily seen, is the necessary condition for reducing the axis $\mathbf{e}_2$ to the axis $\mathbf{i}_2$.

V. The questions or orientation in the rotating coordinate system, which has been considered in this paragraph, have been related to the case when the rotation of the reference basis I has been accomplished about one of its axes (in this case, about the axis $\mathbf{i}_3$). The essential peculiarity of orientation in a rotating coordinate system is the possibility cited in this paragraph of accomplishing orientation with incomplete information from the angular position sensors. This is caused by the fact that the possible equilibrium positions are significantly changed with rotation.

We now consider the case of rigid body orientation in the reference basis I, which is rotating such that the angular velocity vector ω_0 has an arbitrary position in the basis I.

In practice, this case can correspond, for example, to orientation in a coordinate system turned relative to the orbital system by arbitrary (programmed) angles. In this case, the rotation of the reference coordinate system occurs with the same orbital angular velocity ω_0. However, this velocity can have an arbitrary given direction in the basis I.

We will assume that the vector ω_0 is given by three projections in the basis I, i.e. :

$$\omega_0 = \omega_{0I} = \omega_1 i_1 + \omega_2 i_2 + \omega_3 i_3. \qquad (5,46)$$

The rigid body coordinate system E is superimposed with the reference by specifying the correction angular velocity Ω_* and, in addition, the additional velocity ω^*. The additional velocity ω^* must be such that the velocity ω^* coincides with ω_0 when the coordinate systems E and I coincide, i.e., the orientation position is the equilibrium position. The correction Ω_* and ω^* / 223
angular velocities are specified in the body coordinate system

by the components:

$$\left.\begin{aligned}\Omega_{\kappa}=\Omega_{\kappa E}=\Omega_{\kappa 1}e_1+\Omega_{\kappa 2}e_2+\Omega_{\kappa 3}e_3,\\ \omega^{*}=\omega_E^{*}=\omega_1e_1+\omega_2e_2+\omega_3e_3.\end{aligned}\right\} \tag{5.47}$$

The vector sum of these velocities determines the absolute angular velocity of motion of the basis E. The Equations (5.23) will be the equations for the relative motion of the bases E and I, with the value ω_{0I} determined from (5.46), and $\Omega_E=\Omega_{\kappa E}+\omega_E^{*}$ from relations (5.47).

We write these equations, which determine the variation of the components of the quaternion Λ, in scalar form:

$$\left.\begin{aligned}2\dot{\lambda}_0&=-\lambda_1\Omega_{\kappa 1}-\lambda_2\Omega_{\kappa 2}-\lambda_3\Omega_{\kappa 3},\\ 2\dot{\lambda}_1&=\lambda_0\Omega_{\kappa 1}-\lambda_3(\Omega_{\kappa 2}+2\omega_2)+\lambda_2(\Omega_{\kappa 3}+2\omega_3),\\ 2\dot{\lambda}_2&=\lambda_0\Omega_{\kappa 2}-\lambda_1(\Omega_{\kappa 3}+2\omega_{\quad}+\lambda_3(\Omega_{\kappa 1}+2\omega_1),\\ 2\dot{\lambda}_3&=\lambda_0\Omega_{\kappa 3}-\lambda_2(\Omega_{\kappa 1}+2\omega_{1,}\quad{}_1(\Omega_{\kappa 2}+2\omega_2).\end{aligned}\right\} \tag{5.48}$$

The equilibrium positions, which arise as the result of completing the correction process, will be described by the equations:

$$\left.\begin{aligned}\dot{\lambda}_0&=0,\\ \dot{\lambda}_1&=\lambda_2\omega_3-\lambda_3\omega_2,\\ \dot{\lambda}_2&=\lambda_3\omega_1-\lambda_1\omega_3,\\ \dot{\lambda}_3&=\lambda_1\omega_2-\lambda_2\omega_1.\end{aligned}\right\} \tag{5.49}$$

which are obtained from (5.48) with $\Omega_{\kappa i}=0$. The equations (5.49) for constant ω_i $(i=1, 2, 3)$ have the following solutions, which describe the precession rotation:

$$\left.\begin{aligned}\lambda_0(t)&=\lambda_0^0,\\ \lambda_1(t)&=\lambda_1^0\frac{\omega_1^2+(\omega_2^2+\omega_3^2)\cos\omega t}{\omega^2}+\\ &+\lambda_2^0\left[\frac{\omega_3}{\omega}\sin\omega t+\frac{\omega_1\omega_2}{\omega^2}(1-\cos\omega t)\right]-\\ &-\lambda_3^0\left[\frac{\omega_1\omega_3}{\omega^2}(1-\cos\omega t)+\frac{\omega_2}{\omega}\sin\omega t\right].\end{aligned}\right\} \tag{5.50}$$

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$$\left.\begin{aligned}
\lambda_2(t) &= \lambda_1^0\left[\frac{\omega_1\omega_2}{\omega^2}(1-\cos\omega t)-\frac{\omega_3}{\omega}\sin\omega t\right]+\\
&+\lambda_2^0\frac{\omega_2^2+(\omega_1^2+\omega_3^2)\cos\omega t}{\omega^2}+\\
&+\lambda_3^0\left[\frac{\omega_1}{\omega}\sin\omega t+\frac{\omega_2\omega_3}{\omega^2}(1-\cos\omega t)\right].\\
\lambda_3(t) &= \lambda_1^0\left[\frac{\omega_1\omega_3}{\omega^2}(1-\cos\omega t)+\frac{\omega_2}{\omega}\sin\omega t\right]+\\
&+\lambda_2^0\left[\frac{\omega_2\omega_3}{\omega^2}(1-\cos\omega t)-\frac{\omega_1}{\omega}\sin\omega t\right]+\\
&+\lambda_3^0\frac{\omega_3^2+(\omega_1^2+\omega_2^2)\cos\omega t}{\omega^2}.
\end{aligned}\right\} \qquad (5.50)$$

where $\omega^2=\omega_1^2+\omega_2^2+\omega_3^2$. Obviously, the orientation position $\lambda_1=\lambda_2=\lambda_3=0$ is the equilibrium position of the system (5.48).

To investigate the possibility of accomplishing orientation, we will find the Lyapunov function in the form (5.4). Then its derivative composed in accordance with Equations (5.48) will have the form (5.5). Hence, it follows at once that control by the quaternion components of the linear type (5.6) and the arbitrary nonlinear control (5.7), which satisfies the conditions (5.8), ensure asymptotic convergence of the correction process.

We consider the question of possible accomplishment of orientation with incomplete information from the sensors. For the present, we will put aside the question about the possible execution of such a case. Since the angular position sensor is linked with the sighting direction, then obtaining angular mismatch signals is not possible for every position. Thus, with orbital orientation, the sensor is linked to the direction toward the planet and can operate only if its sighting axis is located in a definite region relative to the direction toward the planet. For the present, we will state abstractly that obtairing angular mismatch signals is possible with any position of the reference basis. We will consider two cases: 1) all the rotational angular

components differ from zero; 2) two components of ω_0 do not equal/ 225 zero (the case of one component of ω_0 not equaling zero was considered previously).

1) $\omega_1 \neq 0,\ \omega_2 \neq 0,\ \omega_3 \neq 0$. In this case, accomplishment of the orientation is possible, in principle, only from any one signal from the angular position sensor. Actually, let there be correction only from one channel, for example Ω_{K1}. In this case, the derivative of the Lyapunov function (5.5) will be negative everywhere, except the set $\lambda_1 \equiv 0$ (we assume that the equation is redefined at the point $\lambda_0 = 0$ such that it does not vanish).

The solutions (5.50) show that only one equilibrium position $\lambda_1^0 = \lambda_2^0 = \lambda_3^0 = 0$, i.e., the oriented position, can satisfy the set $\lambda_1 \equiv 0$. These solutions also illustrate the physical meaning of such an orientation: if the quantity $\lambda_1(t)$ is measured, then three measurements at various instants of time are sufficient to determine the quantities λ_1^0, λ_2^0 and λ_3^0 (although these values can also be determined with a simultaneous measurement of the three quantities λ_1, λ_2 and λ_3).

Execution of this case is impossible in practice, in view of the fact that angular position sensors do not exist for such a case of orientation. Thus, for orbital orientation, a programmed position of the reference basis I relative to the orbital axes corresponds to this case. Then nothing physical corresponds to the directions of the reference basis, i.e., they are not directed toward the planet with which the orbit is linked; thus, construction of the control signals is impossible.

2) $\omega_1 = 0,\ \omega_2 \neq 0,\ \omega_3 \neq 0$. It is quite clear that the equations are symmetric and the case of the other components equaling zero reduces to this case. In this case, accomplishment of the

orientation is possible with the use of only one signal from the angular position sensor by the channel Ω_{k2} or Ω_{k3}. Accordingly, accomplishment of orientation is possible from any pair of signals and orientation with one signal Ω_{k1} is impossible.

For proof, we turn to the derivative of the Lyapunov function (5.5), which will be negative everywhere, except for some set determined by the control signal used. Thus, with correction by / 226
the signal Ω_{k2}, it is necessary to verify whether or not the set $\lambda_2 \equiv 0$ contains entire trajectories. One can see from the relations (5.50) with $\omega_1 = 0$ that $\lambda_2 \equiv 0$ is possible if and only if $\lambda_1^0 = \lambda_2^0 = \lambda_3^0 = 0$. Hence, it follows that, with correction by the signal Ω_{k2}, the process of stable orientation occurs. An analogous case occurs with correction by the signal Ω_{k3}. However, with orientation by the signal Ω_{k1}, in accordance with the solutions (5.50), the set $\lambda_1 \equiv 0$ can contain another solution besides the oriented position:

$$\lambda_2^0\omega_3 - \lambda_3^0\omega_2 = 0.$$

Hence, it follows that λ_2^0 and λ_3^0 can be chosen not equal to zero, such that $\lambda_2(t) \neq 0$ and $\lambda_3(t) \neq 0$, although $\lambda_1(t) \equiv 0$.

In practice, this case can correspond, for example, to orientation in the coordinate system I with a programmed turn relative to the orbital axes, if the rotation is performed, for example, in the plane of the local horizon. In this case, the angular velocity of the orbital rotation ω_0 is projected onto two axes of the basis I. On the other hand, since one axis of the basis remains linked to the direction toward the planet, it is possible to measure the deviation of the body basis E relative to the reference In accordance with the proof, accomplishment of the orientation is possible, in this case, with the use of one signal from the angular position sensor (i.e., the signal from

the pitch or bank channel, but not the signal from the course channel). An analytic investigation of the reduction process is complicated in this case, and is possible only in the region of small deviations from the oriented position.

§ 5.4. DYNAMIC ORIENTATION PROBLEM

The conditions of kinematic stability are also the necessary conditions for dynamic stability. An additional condition of dynamic stability is, in particular, the presence in the equation of a damping term, which contains the angular velocity, with a definite dependence of the control on the "kinematic" control function.

We consider the example of dynamic control when the controlling function is shaped linearly. Let the rigid body motion be described by the dynamic Euler equations: / 227

$$\left.\begin{aligned} J_1\dot{\omega}_1 + (J_2 - J_3)\,\omega_2\omega_3 = M_1, \\ J_2\dot{\omega}_2 + (J_3 - J_1)\,\omega_3\omega_1 = M_2, \\ J_3\dot{\omega}_3 + (J_1 - J_2)\,\omega_1\omega_2 = M_3, \end{aligned}\right\} \tag{5.51}$$

where J_1, J_2, J_3 are the principal central moments of inertia of the body and M_1, M_2, M_3 are the external (applied on the body) controlling torques.

The Equations (5.51), along with the kinematic equations, for example, (5.3) completely describe the motion of the controlled body for the state of inertial orientation. The controlling torques M_i (i = 1, 2, 3) should be specified as functions of the angular velocities ω_i of the body and the kinematic parameters, which determine the deviation from the orientation position.

We consider the case when the kinematic equation is specified in the form (5.6). We will specify the control by the linear function:

$$M_i = -h_i\omega_i - k_i\lambda_0\lambda_i. \qquad (5.52)$$

To investigate the stability of such control, we will find the Lyapunov function in the form:

$$T = V + W,$$

where W is the "kinematic" function (5.4), and the term V is determined by the "dynamics" of the body motion:

$$V = \frac{1}{2}(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2) \qquad (5.53)$$

and equals the kinetic energy of its motion. Obviously, the function T is positive-definite and has one zero at the oriented position relative to the inertial basis:

$$\omega_i \equiv 0,\ \lambda_i \equiv 0,\ \lambda_0 = 1 \qquad (i = 1,2,3). \qquad (5.54)$$

The derivative of this function composed in view of Equations (5.51) and (5.3) will equal / 228

$$\dot{T} = \sum_{i=1}^{3} J_i\omega_i M_i + \sum_{i=1}^{3} \lambda_0\lambda_i\omega_i.$$

By inserting the control (5.52), we obtain

$$\dot{T} = -\sum_{i=1}^{3} h_i J_i\omega_i^2 - \sum_{i=1}^{3} \lambda_0\lambda_i\omega_i(k_i - J_i).$$

In the simplest case, one can set $k_i = J_i$ and obtain the negative-definite function $\dot{T}$, which vanishes only at the required orientation position. Hence, it follows that control of the type (5.52) can ensure accomplishment of the orientation.

Rigid body orientation can be accomplished with more general control functions. In particular, we consider control of the type

$$M_i = -F_i(\omega_i - \Omega_{\kappa i}) \quad (i = 1, 2, 3), \tag{5.55}$$

where the functions F_i belong to the class (5.8) and the argument of the control function is the difference of the angular velocity of the rigid body motion and the "kinematic" reduction velocity, which is shaped from the signals from the angular position sensors. The control (5.55) also ensures stable rigid body orientation.

Let us consider the expression (5.53) for the kinetic energy of the rigid body rotation. The derivative $\dot{V}$ composed in accordance with Equations (5.51) and with the control (5.55) will equal:

$$\dot{V} = -\sum_{i=1}^{3} J_i \omega_i F_i(\omega_i - \Omega_{\kappa i}).$$

Obviously, this function is negative everywhere when $|\omega_i| > |\Omega_{\kappa i}|$, and positive when $|\omega_i| < |\Omega_{\kappa i}|$. Hence, it follows that the control (5.55), for large angular velocities, ensures a reduction of the body kinetic energy to a magnitude corresponding to:

$$\omega_i \equiv \Omega_{\kappa i},$$

and, for small velocities, increases it.

Thus, the control (5.55) tends to reduce the motion to / 229
rotation with the angular velocities $\Omega_{\kappa i}$. Under sufficiently rigid controls, such motion can, in practice, coincide with the investigated cases of orientation. Nonetheless, the control (5.55) does not always ensure asymptotic convergence of the process to zero. The form of the function determines the nature of the reduction of the system to zero. Thus, nonlinear control functions can cause natural oscillations of the system about the oriented position.

The study of rigid body motion for various control laws is a separate and very extensive area of investigation, and we will limit ourselves to consideration of examples.

§ 5.5. EXAMPLES

We will assume that the control of the rigid body motion is shaped with the help of angular position signals such that the stable orientation at the equilibrium position is ensured. Then, the control can be specified, for example, in one of the forms described in the preceding paragraphs. It is important that, for orientation in some given coordinate system, the magnitudes of the components of the quaternion, which specifies the transistions from the given coordinate system to the body system, are used as angular position signals. Subsequently, this quaternion will be called the mismatch quaternion.

We will consider the various cases of orientation in different coordinate systems for which the mismatch quaternion is determined by various means. An important application of this is the construction of inertial control systems with the use of the principle of Cardanless systems.

I. Let the position of the reference inertial coordinate system be determined by integration of the kinematic equations. Then the angular velocity is measured with the help of angular velocity sensors rigidly fastened in the body axes. Without decreasing the generality, we will assume that the response axes of the angular velocity sensors are arranged along the axes of the body basis **E**. In this case, the sensors measure the components of the angular velocity vector ω_{ε}, and the kinematic equation (3.11) must be used for integration. / 230

Let the integration begin at the time t_0 and the quaternion (1, 0, 0, 0) be used as the initial quaternion. In this case, the quaternion $\Lambda(t)$ obtained as a result of the integration will describe the motion of the basis E relative to the inertial basis

I with the initial condition $\mathbf{I} = \mathbf{E}(t_0)$. Obviously, if $\Lambda(t)$ is used as the mismatch quaternion for shaping the angular control signal [analogous to (5.6)], then with such control, the orientation will be accomplished such that the body axes are superimposed with the corresponding axes of the reference basis $\mathbf{I} = \mathbf{E}(t_0)$. A block diagram of a similar control system is presented in Figure 5.4, where the closed circuit of an automatic control loop is shown. When considered similar circuits below, we will not depict the closed control loop, assuming that it is always present (we will not depict the "control shaper" and "controlled object" blocks). Our problem involves obtaining the mismatched quaternion, which is the necessary condition for orientation in the given coordinate system.

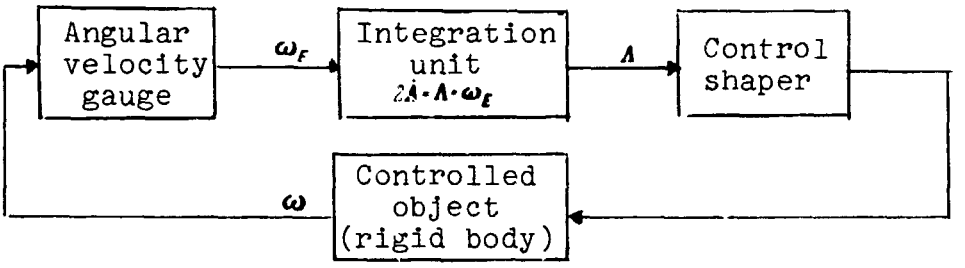

Figure 5.4.

We will now consider the orientation problem, not in the basis $\mathbf{I} = \mathbf{E}(t_0)$, but in another inertial basis, which we denote by $\boldsymbol{E}_\Pi$. Let the transition from the basis I to the basis $\boldsymbol{E}_\Pi$ be determined by the quaternion Λ_Π (Figure 5.5). In this case, it is necessary to use for control the quaternion specifying the position of the basis E relative to the basis $\boldsymbol{E}_\Pi$, i.e., the quaternion Λ_p. Obviously, the following relation is satisfied: / 231

$$\Lambda = \Lambda_\Pi \circ \Lambda_p,$$

from which

$$\Lambda_p = \bar{\Lambda}_\Pi \circ \Lambda. \tag{5.56}$$

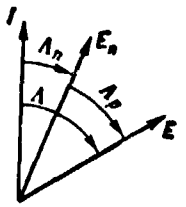

Figure 5.5.

The basis **E** will be superimposed with the given basis Λ_p with control by the quaternion E_n (5.56).

Another method for accomplishing orientation in the basis E_n is possible. Obviously, if the quaternion $\tilde{\Lambda}_n$ is used as the initial condition at the beginning of the integration, then the quaternion obtained as the result of the integration, in accordance with theorem 3.1, will be the quaternion (5.56), i.e., Λ_p will also satisfy the kinematic equation. Hence, it follows that orientation relative to the basis E_n is obtained with this method.

Further, let the orientation be accomplished relative to the basis $\mathbf{I} = \mathbf{E}(t_0)$ and, at some time t, a transition to orientation in the basis E_n is required. To accomplish such a transition, it is necessary to use the quaternion $\Lambda(t)$ (5.56) for control, in place of the quaternion $\Lambda_p(t)$. Such a shift of control quaternion is caused by the required reorientation of the object. Thus, the control system, in which the kinematic equations are integrated, permit one to obtain orientation relative to any inertial basis. The transition from one basis to another is then accomplished by performing an additional multiplication operation (5.56) and by using the results as the mismatch quaternion. The block diagram for such orientation is presented in Figure 5.6.

The transition from orientation in the basis **I** to orientation in another basis **E** can also be accomplished by another method. Let the quaternion obtained as a result of integrating the kinematic equations equal $\Lambda(t_1)$ at the transition time t_1. Obviously, with the integration of the equations beginning at the time t_1, to accomplish orientation in the basis $\mathbf{I} = \mathbf{E}(t_0)$, it is necessary / 232 to take $\Lambda(t_1)$ as the initial condition. If the orientation were accomplished relative to the basis E_n, then, at time t_1, the

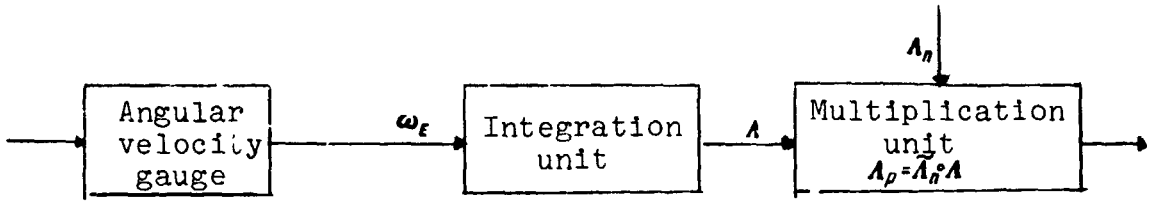

Figure 5.6.

magnitude of the quaternion would equal $\Lambda_p(t_1)=\tilde{\Lambda}_n\circ\Lambda(t_1)$ for the same position of the basis $\mathbf{E}(t_1)$ (cf., Figure 5.5). Hence, it follows that, for the transition at time t_1 from the orientation relative to the basis I to orientation in the basis E_n, one can convert at this time from $\Lambda(t_1)$ to $\tilde{\Lambda}_n\circ\Lambda(t_1)$ in the integration unit. Then one can use the block diagram of the control system shown in Figure 5.4.

II. We consider the problem of orientation relative to a rotating coordinate system. An inertial basis Z, a basis I rotating relative to it and the body basis E are shown in Figure 5.7. Obviously, by analogy to the preceding case, it is necessary to generate the mismatch quaternion for orientation in the basis $\Lambda(t)$ in the following form:

$$\Lambda_p=\tilde{\mathbf{M}}\circ\Lambda. \tag{5.57}$$

The quaternion $\Lambda(t)$ is obtained as a result of integrating the kinematic equations on the basis of measuring the angular velocity ω_E with the corresponding specification of initial conditions. The quaternion M can be obtained (if it is not known beforehand for every moment of time) from the magnitude of the rotational angular velocity ω_0 of the basis T by integrating the kinematic equations with the necessary selection of the initial conditions. Then any form of the kinematic equation [(3.11) or (3.12)] can be used, the choice can be made depending on which basis ω_0 has the simplest expression. The block diagram of such orientation is presented in Figure 5.8. / 233

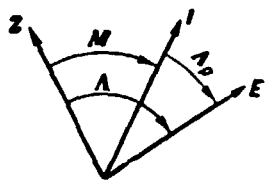

Figure 5.7.

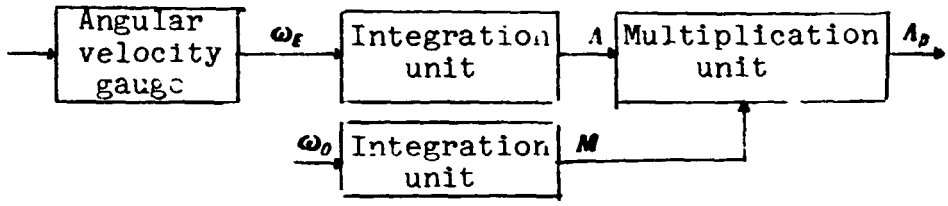

Figure 5.8.

Such orientation can be obtained in another fashion. We note that the required quaternion Λ_p satisfies the kinematic equation [cf., Equation (5.23)]:

$$2\dot{\Lambda}_p = \Lambda_p \circ \omega_E - \omega_{0I} \circ \Lambda_p, \tag{5.58}$$

obtained from (5.57) with consideration of

$$2\dot{\mathbf{M}} = \omega_{0Z} \circ \mathbf{M} = \mathbf{M} \circ \omega_{0I}.$$

The vector ω_E (having as its components the projections of ω onto the body axes) is the measured angular velocity vector. In addition, the rotation vector ω_0 of the fixed basis I can be specified in projections onto the axes of this basis (and not the basis E). In connection with this, one can provide an integration unit executing Equation (5.58). In this case, the block diagram for orientation in a rotating basis can be the diagram (cf., Figure 5.4) with this integration unit. One should note that, in several cases, the diagram presented in Figure 5.8 can be more preferable, since the positions of the two coordinate systems I and Z are maintained continuously in this case, and it is possible to accomplish orientation in either of them.

Apart from orientation directly in the basis I, the presented diagram permits one to accomplish orientation in the basis E_n / 234 having a fixed specified (programmed) position relative to I, which is characterized by the quaternion Λ_n. Let the relative position of the bases Z, I, E_n and E be determined by the quaternions

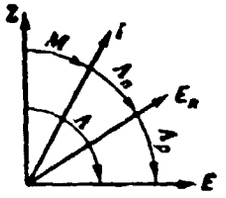

Figure 5.9.

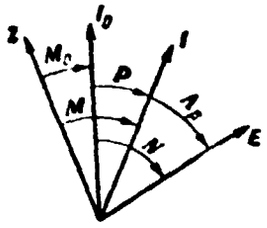

Figure 5.10.

$\mathbf{M}$, Λ_n, Λ_p and Λ in accordance with Figure 5.9. Obviously, the quaternion:

$$\Lambda_p = \tilde{\Lambda}_n \circ \tilde{\mathbf{M}} \circ \Lambda. \tag{5.59}$$

should be used for control for orientation relative to the basis E_n. The block diagram of such orientation is presented in Figure 5.8, which includes another multiplication unit performing the multiplication of the quaternion $\tilde{\Lambda}_n$ by the quaternion $\tilde{\mathbf{M}} \circ \Lambda$.

Since the quaternion Λ_p (5.59) satisfies (for a constant $\tilde{\Lambda}_n$) Equation (5.58), then the same type of orientation can also be obtained in the block diagram in Figure 5.4 with the use of an integration unit executing Equation (5.58). For the transition from orientation in the basis I to orientation in the basis E_n at the time t_1, it is necessary to produce in the integrator a change of the initial conditions: one should write the quantity $\Lambda_p(t_1)$ in place of the quaternion $\tilde{\Lambda}_n \circ \Lambda_p(t_1)$ obtained as the result of integration. Then, as follows from relations (5.57) and (5.58), the required magnitude of the mismatch quaternion (5.59) will be obtained.

III. In investigating the methods for accomplishing orientation in a rotating basis it was assumed that the quaternions $\Lambda(t)$ and $\mathbf{M}(t)$, which are obtained as a result of integrating the kinematic equations, or the quaternion $\Lambda_p(t)$ itself have correspondingly selected initial conditions, which ensure the

required orientation of the basis **I** (and **Z**). However, this does not always occur in practice, and the basis **I**, in which the orientation is carried out, does not coincide with the required basis $\mathbf{I}_0$ (because of drift and errors in the initial conditions).

Let us assume that an angular position sensor is fastened to the controlled object, i.e., to the basis **E**, which determines the orientation of the basis **E** relative to the basis $\mathbf{I}_0$. Without decreasing the generality, we will assume that the sensor measures the components of the quaternion **N**(t), which determines the transition from the basis $\mathbf{I}_0$ to the basis **E** (Figure 5.10). Let the mismatch of the bases $\mathbf{I}_0$ and **I** be determined by the quaternion **P**. Obviously, it is possible to determine **P** from the available quaternions Λ_p and **N** / 235

$$\mathbf{P} = \mathbf{N} \circ \tilde{\Lambda}_p. \tag{5.60}$$

The motion of the basis **I** relative to the inertial basis **Z** is specified by the quaternion **M** and occurs with the angular velocity $\boldsymbol{\omega}_0^*$; the same motion of the basis $\mathbf{I}_0$ relative to the basis **Z** occurs with the velocity ω_0 and is specified by the quaternion $\mathbf{M}_0$. Then the problem of investigating the relative motion of the bases **I** and $\mathbf{I}_0$ completely coincides with the problem considered in Paragraph 5.3 of orientation in a rotating coordinate system. Actually, we have, for the quaternion **P**, in accordance with Figure 5.10:

$$\mathbf{P} = \tilde{\mathbf{M}}_0 \circ \mathbf{M},$$

from which we obtain the equation of the relative motion:

$$2\dot{\mathbf{P}} = -2\tilde{\mathbf{M}}_0 \circ \dot{\mathbf{M}}_0 \circ \tilde{\mathbf{M}}_0 \circ \mathbf{M} + 2\tilde{\mathbf{M}}_0 \circ \dot{\mathbf{M}} = -\boldsymbol{\omega}_{0I_0} \circ \mathbf{P} + \mathbf{P} \circ \boldsymbol{\omega}_{0I}^*, \tag{5.61}$$

where, in accordance with the condition presented above:

$$\left.\begin{aligned} 2\dot{\mathbf{M}}_0 &= \omega_{0Z} \circ \mathbf{M}_0 = \mathbf{M}_0 \circ \omega_{0I_0}, \\ 2\dot{\mathbf{M}} &= \omega^*_{0Z} \circ \mathbf{M} = \mathbf{M} \circ \omega^*_{0I}. \end{aligned}\right\} \tag{5.62}$$

For reducing the basis **I** to the basis $\mathbf{I}_0$, it is necessary to introduce an additional correction angular velocity $\Omega_к$ of the motion of the basis **I**, i.e., the total angular velocity of the basis **I** must equal:

$$\Omega_к + \omega^*_0.$$

In this case, Equation (5.61) changes into the equation of the corrected coordinate system:

$$2\dot{\mathbf{P}} = -\omega_{0I_0} \circ \mathbf{P} + \mathbf{P} \circ (\omega^*_{0I} + \Omega_{кI}), \tag{5.63}$$

analogous to Equation (5.23). Accordingly, the second equation of/ 236 (5.62), which determines the motion of the rotating basis **I** relative to the inertial basis **Z**, changes into the equation:

$$2\dot{\mathbf{M}} = \mathbf{M} \circ (\omega^*_{0I} + \Omega_{кI}). \tag{5.64}$$

The execution of this equation should also be accomplished to serve the problem of correcting the basis **I**. The available (calculated) quaternion (5.60) can be used to shape the correction signal. All the correction methods considered in Paragraph 5.3 then ensure a stable process for reducing the basis **I** to the basis $\mathbf{I}_0$. The block diagram of such a control process with position correction of the basis **I** is presented in Figure 5.11. As is seen in the figure, this diagram has two control loops: the rigid body position control loop and the correction loop, which reduces the basis **I** to the required position.

We note, in conclusion, that the angular position sensor does not necessarily have to have a linear characteristic in the function of the components of the quaternion **N** and to measure the

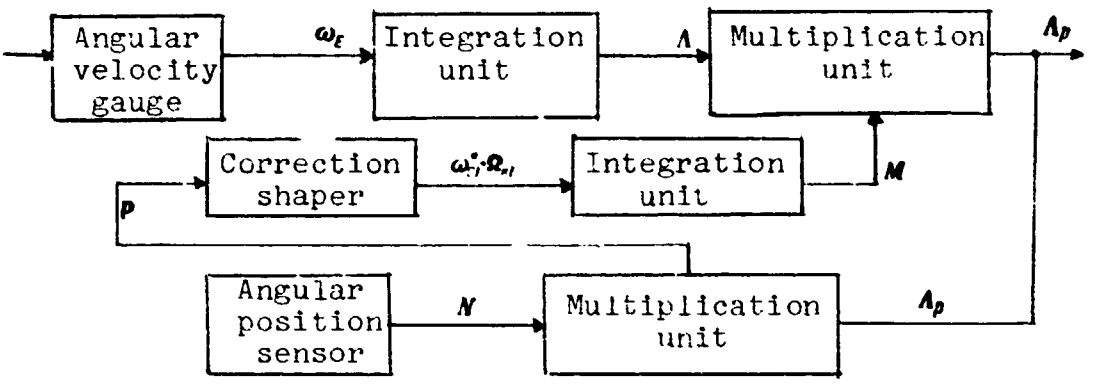

Figure 5.11.

quantity **N** for any position of the basis **E**. With actual operation of the control loop, the magnitude of the mismatch $\tilde{\Lambda}_p$ will be small, and, for small deviations of the basis **I** from the basis I_0, the quaternion **P** will also be close to one. For small deviations, the formula (5.60) changes into the vector form for adding the small vectors of the angular mismatches. It can be shown that a rather wide class of functional dependences of the angular position output signals can be used for correction. However, the investigation of this question is rather complicated and is specialized in nature. / 237

The presented diagrams for constructing a control system can be used, for example, for orientating spacecraft in an orbital coordinate system with its flight about a planet. Sensors, which determine the direction toward the center of the planet or the direction of the spacecraft velocity vector, can serve as angular position sensors.

IV. In a majority of practical cases of inertial orientation, an arbitrary position of the inertial basis I is not required, but a definite position, which is linked, for example, to a celestial coordinate system (equatorial, ecliptic, etc.). We denote this required position of the coordinate system by I_0.

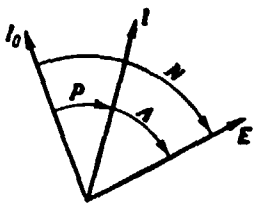

Figure 5.12.

Obviously, for the basis **I** to be superimposed with the basis $\mathbf{I}_0$, it is necessary to begin the integration of the kinematic equations in the integration unit (cf., Figure 5.4) at the moment when the basis **E** coincides with the basis $\mathbf{I}_0$.

However, there is another possibility for the superposition of the bases **I** and $\mathbf{I}_0$, which becomes necessary because of the drifts of the basis **I** when integrating the kinematic equations, and also when it is impossible to advance the basis **I** to the position of $\mathbf{I}_0$. Let stellar sensors, which permit one to determine the position of the basis **E** relative to the basis $\mathbf{I}_0$, be established in the basis **E**. Without specifying the type of these sensors, we will assume that they measure the components of the quaternion **N** (Figure 5.12).

Since the value of the quaternion $\mathbf{\Lambda}(t)$ is calculated in the integration unit and is thus a known quantity, then it is possible to calculate the quaternion **P**, which characterizes the mismatch of the bases **I** and $\mathbf{I}_0$, from the relation (5.60). In view of the fact that the bases **I** and $\mathbf{I}_0$ are inertial, the quaternion **P** remains constant. We introduce the correction angular velocity $\mathbf{\Omega}_{\text{к}}$, which we will add to the velocity $\boldsymbol{\omega}_{\varepsilon}$ at the input of the integration unit, and specify the quantity $\mathbf{\Omega}_{\text{к}}$ as a function of the quaternion **P**.

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The motion of the basis **E** relative to the basis $\mathbf{I}_0$ proceeds with the same angular velocity $\boldsymbol{\omega}_{\varepsilon}$ (the absolute angular velocity of the motion of the basis **E**); thus, we have the equation for the change of **N**:

$$2\dot{\mathbf{N}} = \boldsymbol{\omega}_{I_0} \circ \mathbf{N} = \mathbf{N} \circ \boldsymbol{\omega}_{\varepsilon}.$$

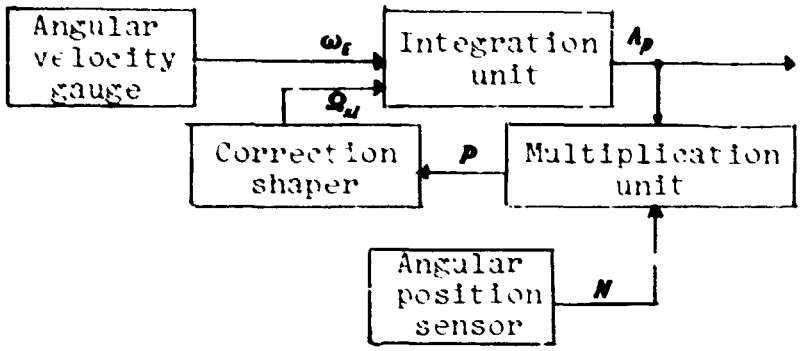

Figure 5.13.

The equation, which is executed in the integration unit with the correction taken into account, will be

$$2\dot{\Lambda}_p = (\omega_I + \Omega_{xI}) \circ \Lambda_p = \Lambda_p \circ (\omega_E + \Omega_{xE}) = \Lambda_p \circ \omega_E + \Omega_{xI} \circ \Lambda_p. \qquad (5.65)$$

Hence, in accordance with (5.60), we obtain the following equation, which describes the motion of the basis **I** relative to / 239 the basis $\mathbf{I}_0$:

$$2\dot{\mathbf{P}} = 2\dot{\mathbf{N}} \circ \tilde{\Lambda}_p - 2\mathbf{N} \circ \tilde{\Lambda}_p \circ \dot{\Lambda}_p \circ \tilde{\Lambda}_p = \\ = \omega_{I_0} \circ \mathbf{P} - \mathbf{P} \circ (\omega_I + \Omega_{xI}) = -\mathbf{P} \circ \Omega_{xI}, \qquad (5.66)$$

since $\omega_{I_0} \circ \mathbf{P} = \mathbf{P} \circ \omega_I$. Equation (5.66) indicates that the process of correcting the position of the basis **I** is identical to the case considered in Paragraph 5.2 of orientation in an inertial coordinate system.

The block diagram for orientation in the corrected system **I** is presented in Figure 5.13. The integration unit, in this case, should execute the equation in the form (5.65). The system also contains loops for orientation and correction of the reference basis. We note that generating the magnitude of the correction angular velocity from the stellar sensor signals is a special problem. The complexity of this problem involves the fact that the stellar sensors have limited output characteristics, their output signals characterize the vector of the angular

mismatch in arbitrary axes, which are determined by the arrangement of the sighting stars. It is possible to accomplish correction with incomplete and excess information from the sensors. This problem also requires special investigation.

The examples considered here indicate that completion of the orientation of various types can be accomplished in the diagrams using two standard operations: integration of the kinematic equations and quaternion multiplication.

CHAPTER 6

OPTIMUM SPATIAL TURNS OF A RIGID BODY / 240

The problem of an optimum spatial turn of a rigid body has received practical interest with the investigation of problems of spacecraft control. Up to the present time, there has been a number of works devoted to this problem, in which some particular cases have been solved. Among these are the optimization of a plane turn about one of the principal axes of inertia of the body and optimization of the turn control of a rapidly rotating body in space [18, 50, 61].

The problem of an optimum turn, in our view, is one of the most complicated problems of theoretical mechanics. The formalism of the theory of optimum control leads, in the general case, either to linear differential equations with variable coefficients, or to nonlinear equations, whose solution cannot be obtained in general form. The problem of constructing optimum control of a spatial turn as a function of the phase coordinates and velocities, i.e., the problem of synthesizing optimum control, is even further from completion.

The problem of optimizing a spatial turn of a rigid body with respect to the speed of response will be considered in this chapter. The problem of the so-called kinematic turn, where the magnitudes of the angular velocity components of the rigid body are the control, will be investigated in the first four paragraphs. This problem is completely solved. It is shown that control by quaternions is close to optimum. This fact gives a great advantage

to the control, which is constructed from the components of the mismatch quaternion.

The problem of the dynamic optimum turn is considered in the next two paratgraphs in a formulation of optimizing the turn with respect to the speed of response and with respect to the fuel consumption in the turn. The solution of the problem is given for the particular case of control of a rigid body with spherical symmetry.

§ 6.1. KINEMATIC OPTIMUM TURN. FORMULATION OF THE PROBLEM / 241

By the kinematic problem of a spatial turn of a rigid body, we will mean the problem of controlling the motion of rigid body, when the controlling function is the magnitude of the angular velocity of the rigid body motion. In this case, the equations of motion of the controlled object (i.e., the rigid body) will be the kinematic equations, in which the angular velocity components are the control function. Obviously, such a formulation of the problem does not take into account the fact that the motion of a rigid body is actually described by the dynamic Euler equations (5.51) and that the magnitudes of the torques acting on the body actually emerge as the control function (such is just the formulation of the problem of the dynamic turn of a rigid body). Nonetheless, the kinematic problem of a spatial turn has practical value. Actually, the control system for the spatial orientation of a rigid body (for example, spacecraft) is constructed such that the control is always accomplished by the three components (projections) of the angular velocity of the body.

Let us consider the position of a rigid body described by the body basis **E**. We will specify the motion of the basis **E** relative to a reference basis **I** by the quaternion Λ. This motion

is described by the kinematic Equation (3.11), which we write in quaternion and coordinate forms:

$$\left.\begin{aligned} 2\dot{\Lambda} &= \Lambda \circ \omega_E, \\ 2\dot{\lambda}_0 &= -\lambda_1\omega_1 - \lambda_2\omega_2 - \lambda_3\omega_3, \\ 2\dot{\lambda}_1 &= \lambda_0\omega_1 + \lambda_2\omega_3 - \lambda_3\omega_2, \\ 2\dot{\lambda}_2 &= \lambda_0\omega_2 + \lambda_3\omega_1 - \lambda_1\omega_3, \\ 2\dot{\lambda}_3 &= \lambda_0\omega_3 + \lambda_1\omega_2 - \lambda_2\omega_1, \end{aligned}\right\} \qquad (6.1)$$

where ω_i $(i = 1, 2, 3)$ are the components of the angular velocity of the motion of the basis **E** relative to the basis **I** in the projections onto the axes of the basis **E**.

The quantities ω_i are the control functions in this problem./ 242 We will assume that the set of function $\omega_i(t)$ is bounded. We will denote this set by the symbol Ξ_ε and will consider that it is closed:

$$\omega_i(t) \in \Xi_\varepsilon. \qquad (6.2)$$

We will consider the motion in the time interval $[t_0, T]$. We will specify the initial position of the body by the quaternion Λ_0, and its final position by the quaternion Λ_T, i.e., we have the following boundary conditions on the motion:

$$\Lambda(t_0) = \Lambda_0, \quad \Lambda(T) = \Lambda_T. \qquad (6.3)$$

We will look for the control $\omega_i(t)$, which minimizes the transition time from the initial to the final position. Equations (6.1) together with the conditions (6.2) and (6.3), completely determine the kinematic problem of a spatial turn.

Let us consider the kinematic Equation (3.12). Obviously, the equation in this form has a more limited application, since it reflects the process of control of the rigid body motion when

the components of the angular velocity projections onto the initial reference basis are the control function. Investigation of the second form of the kinematic equation is of greater theoretical interest.

We write the equation of motion (3.12) in quaternion and coordinate forms:

$$\left.\begin{aligned} 2\dot{\Lambda} &= \omega_I \circ \Lambda, \\ 2\dot{\lambda}_0 &= -\lambda_1\Omega_1 - \lambda_2\Omega_2 - \lambda_3\Omega_3, \\ 2\dot{\lambda}_1 &= \lambda_0\Omega_1 + \lambda_3\Omega_2 - \lambda_2\Omega_3, \\ 2\dot{\lambda}_2 &= \lambda_0\Omega_2 + \lambda_1\Omega_3 - \lambda_3\Omega_1, \\ 2\dot{\lambda}_3 &= \lambda_0\Omega_3 + \lambda_2\Omega_1 - \lambda_1\Omega_2, \end{aligned}\right\} \qquad (6.4)$$

where Ω_i $(i = 1, 2, 3)$ are the components of the angular velocity of the motion of the basis **E** relative to the basis **I** in projections onto the axes of the basis **I**.

We will consider the control limitied, i.e., / 243

$$\Omega_i \in \Xi_I, \qquad (6.5)$$

where the set Ξ_I will be considered distinct from the set (6.2). The conditions (6.3) will serve as the boundary conditions of the motion for this case.

Let us proceed to formulation of the necessary conditions for control optimization with respect to the speed of response, for which we will use the formalism of Pontryagin's principle of the maximum [6, 41].

We will introduce the supplementary functions $\psi_0, \psi_1, \psi_2, \psi_3$, corresponding to the phase variables $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, which completely define the motion. We will set up the Hamiltonian for the equations of motion (6.1):

$$\mathcal{H}_E = \langle \dot{\lambda}\psi \rangle = -\frac{1}{2}\psi_0(\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3) + \\ + \frac{1}{2}\psi_1(\lambda_0\omega_1 + \lambda_2\omega_3 - \lambda_3\omega_2) + \frac{1}{2}\psi_2(\lambda_0\omega_2 + \lambda_3\omega_1 - \lambda_1\omega_3) + \\ + \frac{1}{2}\psi_3(\lambda_0\omega_3 + \lambda_1\omega_2 - \lambda_2\omega_1). \tag{6.6}$$

The functions ψ_i $(i = 0, 1, 2, 3)$ must satisfy the so-called conjugate system of equations:

$$\dot{\psi}_i = -\frac{\partial \mathcal{H}_E}{\partial \lambda_i}, \tag{6.7}$$

which, in accordance with (6.6), takes the form

$$\left.\begin{aligned} 2\dot{\psi}_0 &= -\psi_1\omega_1 - \psi_2\omega_2 - \psi_3\omega_3, \\ 2\dot{\psi}_1 &= \psi_0\omega_1 + \psi_2\omega_3 - \psi_3\omega_2, \\ 2\dot{\psi}_2 &= \psi_0\omega_2 + \psi_3\omega_1 - \psi_1\omega_3, \\ 2\dot{\psi}_3 &= \psi_0\omega_3 + \psi_1\omega_2 - \psi_2\omega_1. \end{aligned}\right\} \tag{6.8}$$

To determine the control, we have four scalar equations of motion (6.1), four Equations (6.8), and three conditions for the maximum, which involve the fact that, for optimum control, the function $\mathcal{H}_E$ (6.6) reaches a maximum on the set Ξ_E (6.2):

$$\left.\begin{aligned} \mathcal{H}_E[\lambda_i(t), \psi_i(t), \omega_k(t)] &= \max_{\omega_k \in \Xi_E} \mathcal{H}_E[\psi_i(t), \lambda_i(t), \omega_k(t)] \\ (i &= 0, 1, 2, 3;\ k = 1, 2, 3). \end{aligned}\right\} \tag{6.9}$$

We have eight boundary conditions (6.3) for the eight differential Equations (6.1) and (6.8). Thus, the solutions $\lambda_i(t)$, $\psi_i(t)$ and the control $\omega_k(t)$ must be uniquely determined. However, the conditions (6.9) determine the control ω_k as a function of the quantities ψ_i and λ_i, and Equations (6.1) and (6.8) are linear equations, whose variable coefficients are functions of ω_k. Hence, it is seen that these equations cannot be solved directly (since there is no general solution for linear differential equations with variable coefficients and, in this case, there is also no general solution of the kinematic equations). / 244

For the second form for presenting the problem of an optimum kinematic turn, i.e., for Equations (6.4), we obtain (for the same designations of the supplementary functions ψ_i) the following Hamiltonian:

$$\begin{aligned}\mathcal{H}_I = (\lambda\psi) = -\frac{1}{2}\psi_0(\lambda_1\Omega_1 + \lambda_2\Omega_2 + \lambda_3\Omega_3) + \\ + \frac{1}{2}\psi_1(\lambda_0\Omega_1 + \lambda_3\Omega_2 - \lambda_2\Omega_3) + \frac{1}{2}\psi_2(\lambda_0\Omega_2 + \lambda_1\Omega_3 - \lambda_3\Omega_1) + \\ + \frac{1}{2}\psi_3(\lambda_0\Omega_3 + \lambda_2\Omega_1 - \lambda_1\Omega_2).\end{aligned} \tag{6.10}$$

For this case, the conjugate system of Equations (6.7) takes the form:

$$\left.\begin{aligned}2\dot{\psi}_0 &= -\psi_1\Omega_1 - \psi_2\Omega_2 - \psi_3\Omega_3,\\ 2\dot{\psi}_1 &= \psi_0\Omega_1 + \psi_3\Omega_2 - \psi_2\Omega_3,\\ 2\dot{\psi}_2 &= \psi_0\Omega_2 + \psi_1\Omega_3 - \psi_3\Omega_1,\\ 2\dot{\psi}_3 &= \psi_0\Omega_3 + \psi_2\Omega_1 - \psi_1\Omega_2.\end{aligned}\right\} \tag{6.11}$$

The control Ω_k $(k = 1, 2, 3)$ will be optimum, if the Hamiltonian $\mathcal{H}_I$ reaches a maximum with this control, i.e.,

$$\mathcal{H}_I[\lambda_i(t), \psi_i(t), \Omega_k(t)] = \max_{\Omega_k \in \Sigma_I} \mathcal{H}_I[\lambda_i(t), \psi_i(t), \Omega_k(t)]. \tag{6.12}$$

The condition for the maximum (6.12), together with Equations (6.4) and (6.11) with the boundary conditions (6.3), are sufficient to determine the optimum control and the corresponding motion. However, we have here the same difficulty as in the preceding case: the solution of the kinematic equation for arbitrary functions $\Omega_k(t)$ cannot be obtained directly. / 245

§ 6.2. SOLUTION OF THE KINEMATIC OPTIMUM TURN PROBLEM

The solution of the kinematic optimum turn problem can be obtained, thanks to the fact that the systems of conjugate Equations (6.8) and (6.11) for the supplementary variables ψ_i coincide

with the equations of motion (6.1) and (6.4), respectively.* As is not difficult to see from (6.8), the conjugate system can be written in the form of a quaternion equation (3.11):

$$2\dot{\Psi} = \Psi \circ \omega_E, \tag{6.13}$$

where $\Psi = \psi_0 + \psi_1 i_1 + \psi_2 i_2 + \psi_3 i_3$ is the quaternion defined by the supplementary functions.

In accordance with theorem 3.1, one can state that, since Equations (6.8) and (6.1) are kinematic equations for the same angular velocity ω_E, the functions $\Psi(t)$ and $\Lambda(t)$ must differ by a constant:

$$\Psi = C_E \circ \Lambda, \tag{6.14}$$

where C_E is a constant quaternion. It is not difficult to verify that (6.14) actually occurs. We consider for this quaternion:

$$C_E = \Psi \circ \tilde{\Lambda}$$

and find its derivative:

$$2\dot{C}_E = 2\dot{\Psi} \circ \tilde{\Lambda} - 2\Psi \circ \tilde{\Lambda} \circ \dot{\Lambda} \circ \tilde{\Lambda} =$$
$$= \Psi \circ \omega_E \circ \tilde{\Lambda} - \Psi \circ \tilde{\Lambda} \circ \Lambda \circ \omega_E \circ \tilde{\Lambda} = 0,$$

which actually vanishes.

Let us now consider the Hamiltonian $\mathcal{H}_E$ (6.6), and represent/ 246 it in the following form by separating terms in the control functions:

*One can show that the same result is valid for any linear system of differential equations $\dot{X} = AX$ having an antisymmetric matrix A, and the solution of the optimum control problem can be obtained.

$$\mathcal{H}_E = \frac{1}{2}\omega_1(\psi_1\lambda_0 - \psi_0\lambda_1 + \psi_2\lambda_3 - \psi_3\lambda_2) + \\ + \frac{1}{2}\omega_2(\psi_2\lambda_0 - \psi_0\lambda_2 + \psi_3\lambda_1 - \psi_1\lambda_3) + \\ + \frac{1}{2}\omega_3(\psi_3\lambda_0 - \psi_0\lambda_3 + \psi_1\lambda_2 - \psi_2\lambda_1).$$

We introduce the vector function $\mathbf{p_E}(t)$ (i.e., the quaternion $\mathbf{p_E}$), which we define by its following components:

$$\left.\begin{aligned} p_{1E} &= \psi_1\lambda_0 - \psi_0\lambda_1 + \psi_2\lambda_3 - \psi_3\lambda_2, \\ p_{2E} &= \psi_2\lambda_0 - \psi_0\lambda_2 + \psi_3\lambda_1 - \psi_1\lambda_3, \\ p_{3E} &= \psi_3\lambda_0 - \psi_0\lambda_3 + \psi_1\lambda_2 - \psi_2\lambda_1. \end{aligned}\right\} \qquad (6.15)$$

We will call the vector function $\mathbf{p_E}(t)$ the conjugate control vector. One can see that the Hamiltonian can be written in the form:

$$\mathcal{H}_E = \frac{1}{2}(\omega_1 p_{1E} + \omega_2 p_{2E} + \omega_3 p_{3E}) = \frac{1}{2}\omega_E \cdot p_E, \qquad (6.16)$$

and thus, the condition for the maximum of the function $\mathcal{H}_E$ on the set Ξ_E determines the control ω_E as a function of the conjugate vector. The form of this dependence will be determined by the configuration of the set Ξ_E.

The conjugate control vector is determined by the quantities ψ_i and λ_i with the help of the relations (6.15). It was previously established that the functions $\psi_i(t)$ can be expressed in terms of $\lambda_i(t)$, since Equations (6.8) and (6.1) have the same form and contain the same velocity ω_E. Hence, it follows that it is possible in principle to express $\mathbf{p_E}$ as a function only of $\lambda_i(t)$. Actually, one can easily verify that the equalities (6.15) can be written in the form of the product of quaternions:

$$p_E = \mathrm{vect}(\tilde{\Lambda} \circ \Psi), \qquad (6.17)$$

i.e., the function p_E is the vector part of the product of the quaternions $\tilde{\Lambda}$ and Ψ. However, the relation between the quaternions Ψ and Λ is given by the equality (6.14), from which it follows that the unknown product equals: / 247

$$\tilde{\Lambda} \circ \Psi = \tilde{\Lambda} \circ C_E \circ \Lambda \tag{6.18}$$

and is determined only by the quaternion Λ.

Let us find the scalar part of the quaternion $\tilde{\Lambda} \circ \Psi$:

$$\operatorname{sqal}(\tilde{\Lambda} \circ \Psi) = \operatorname{sqal}(\tilde{\Lambda} \circ C_E \circ \Lambda) = \operatorname{sqal}(\tilde{\Lambda} \circ \Lambda \circ C_E) = \operatorname{sqal} C_E \tag{6.19}$$

(since the norm of the quaternion Λ equals one).

It should also be stated that, in view of the homogeneity of the conjugate system, one can select the quaternion Ψ such that its norm also equals one. Hence, it will follow that the norm of C_E also equals one:

$$\| \tilde{\Lambda} \circ \Psi \| = \| \tilde{\Lambda} \circ C_E \circ \Lambda \| = \| C_E \| = \| \Psi \| \| \Lambda \| = 1.$$

Now, by using the equalities (6.18) and (6.19), one can find the magnitude of the vector part of the product of the quaternions Ψ and Λ:

$$\operatorname{vect}(\tilde{\Lambda} \circ \Psi) = \tilde{\Lambda} \circ \Psi - \operatorname{sqal}(\tilde{\Lambda} \circ \Psi) = \tilde{\Lambda} \circ C_E \circ \Lambda - \operatorname{sqal} C_E =$$
$$= \tilde{\Lambda} \circ (C_E - \operatorname{sqal} C_E) \circ \Lambda = \tilde{\Lambda} \circ (\operatorname{vect} C_E) \circ \Lambda = \tilde{\Lambda} \circ c_E \circ \Lambda.$$

Thus, we finally obtain, according to (6.17), the following basic relation determining the conjugate control vector:

$$p_E = \tilde{\Lambda} \circ c_E \circ \Lambda. \tag{6.20}$$

With this, the kinematic problem of the optimum spatial turn is completely solved. Actually, the condition for the maximum of the function $\mathcal{H}_\varepsilon$ (6.9), with the use of the expression (6.16), determines the control $\omega_\varepsilon(t)$ as a function of the conjugate vector $\mathbf{p_E}$. The vector $\mathbf{p_E}$ is determined as a function of the phase coordinate and some constant vector $\mathbf{c_E}$ by the relation (6.20). After being substituted into the quaternion equation of motion (6.1), the control functions $\omega_\varepsilon(t)$ contribute to the equation of motion the same phase coordinates and constant $\mathbf{c_E}$, / 248 whose selection must uniquely satisfy the boundary (final) conditions of the motion. Thus, in principle (not considering the problem of analytic solvability of the obtained equations), the problem is completely solved.

One should note that the quaternion $\mathbf{\Lambda}$ is a double-valued function of the rigid body position (Λ and $-\Lambda$ are identical to one position). According to Equations (6.8) and (6.13), the quaternion Ψ can also be double-valued. Hence, it follows that it is necessary to set the quaternion $\mathbf{p_E}$, as well as $\mathbf{c_E}$, double-valued, i.e., to allow the possible existence of the two values $+\mathbf{c_E}$ and $-\mathbf{c_E}$, which simultaneously satisfy the solution of the optimum problem. As we will show later, this assumption is very significant.

Let us now find the solution of the optimum turn problem for the second form of the equations. The conjugate system of Equations (6.11) for the supplementary variables ψ_i for this case coincides with the equations of motion (6.4) and, consequently, can be written in the form:

$$2\dot{\Psi} = \omega_I \circ \Psi, \tag{6.21}$$

where Ψ is the quaternion defined above [cf., (6.13)].

In accordance with theorem 3.1, Equations (6.21) and (6.4) have the same fundamental system of solutions and, therefore, the function $\Psi(t)$ differs from $\Lambda(t)$ by a constant quaternion, i.e., the following relation is valid:

$$\Psi = \Lambda \circ C_I. \tag{6.22}$$

Let us make certain that such an equality is actually satisfied. According to (6.22), we have the following expression for the constant

$$C_I = \tilde{\Lambda} \circ \Psi.$$

Its derivative obtained in accordance with Equations (6.4) and (6.21) actually equals zero:

$$\begin{aligned} 2\dot{C}_I &= -2\tilde{\Lambda} \circ \dot{\Lambda} \circ \tilde{\Lambda} \circ \Psi + 2\tilde{\Lambda} \circ \dot{\Psi} = \\ &= -\tilde{\Lambda} \circ \omega_I \circ \Lambda \circ \tilde{\Lambda} \circ \Psi + \tilde{\Lambda} \circ \omega_I \circ \Psi = 0. \end{aligned}$$

Let us consider the Hamiltonian $\mathcal{H}_I$ (6.10) and represent it / 249 in the form:

$$\begin{aligned} \mathcal{H}_I = &\frac{1}{2}\Omega_1(-\lambda_1\psi_0 + \lambda_0\psi_1 + \lambda_2\psi_3 - \lambda_3\psi_2) + \\ &+ \frac{1}{2}\Omega_2(\lambda_0\psi_2 - \lambda_2\psi_0 + \psi_1\lambda_3 - \lambda_1\psi_3) + \\ &+ \frac{1}{2}\Omega_3(\lambda_0\psi_3 - \lambda_3\psi_0 + \lambda_1\psi_2 - \lambda_2\psi_1). \end{aligned}$$

By analogy to (6.15), we introduce the vector function $\mathbf{p}_I(t)$ (i.e., the quaternion $\mathbf{p}_I$), which is defined by the variables λ_i and ψ_i, such that:

$$\left.\begin{aligned} p_{1I} &= \lambda_0\psi_1 - \lambda_1\psi_0 + \lambda_2\psi_3 - \psi_2\lambda_3, \\ p_{2I} &= \lambda_0\psi_2 - \lambda_2\psi_0 + \lambda_3\psi_1 - \lambda_1\psi_3, \\ p_{3I} &= \lambda_0\psi_3 - \lambda_3\psi_0 + \lambda_1\psi_2 - \lambda_2\psi_1. \end{aligned}\right\} \tag{6.23}$$

We will also call the function $\mathbf{p}_I(t)$ the conjugate control vector. With its help, the Hamiltonian can be represented in

the form of a scalar product of the control function ω_I and the conjugate vector $\mathbf{p}_I$:

$$\mathcal{H}_I = \frac{1}{2}(\Omega_1 p_{1I} + \Omega_2 p_{2I} + \Omega_3 p_{3I}) = \frac{1}{2}\omega_I \cdot p_I. \tag{6.24}$$

Since the vector $\mathbf{p}_I$ is determined as a function of Ψ and Λ by the relations (6.23), and the quaternion Ψ is expressed through Λ by the equality (6.22), then it is possible to express $\mathbf{p}_I$ as a function only of the variable $\Lambda(t)$. Actually, it can be verified that the relations (6.23) can be represented in form of a product of quaternions:

$$p_I = \text{vect}(\Psi \circ \tilde{\Lambda}). \tag{6.25}$$

By considering the obtained product, in accordance with (6.22), we obtain:

$$\Psi \circ \tilde{\Lambda} = \Lambda \circ C_I \circ \tilde{\Lambda}.$$

Let us find the scalar part of the quaternion $\Psi \circ \tilde{\Lambda}$:

$$\text{sqal}(\Psi \circ \tilde{\Lambda}) = \text{sqal}(\Lambda \circ C_I \circ \tilde{\Lambda}) = \text{sqal}(\Lambda \circ \tilde{\Lambda} \circ C_I) = \text{sqal}\, C_I.$$

Hence, we obtain the expression for its vector part in the form: / 250

$$\text{vect}(\Psi \circ \tilde{\Lambda}) = \Psi \circ \tilde{\Lambda} - \text{sqal}\, C_I = \\ = \Lambda \circ (C_I - \text{sqal}\, C_I) \circ \tilde{\Lambda} = \Lambda \circ c_I \circ \tilde{\Lambda}.$$

Thus, we finally have the following expression for the conjugate control vector:

$$p_I = \Lambda \circ c_I \circ \tilde{\Lambda}. \tag{6.26}$$

The conjugate control vector (6.26) is expressed as a function only of the phase coordinates and three constants. The conditions for maximum of the function $\mathcal{H}_I$ (6.24) determines the

control ω_j as a function of $\mathbf{p}_I$ and, therefore, the phase coordinates and constants. By substituting this control into the initial equation of motion (6.4), we obtain a closed system of differential equations, which, generally speaking, is always solvable. The constant $\mathbf{c}_I$ should then be chosen in order to satisfy the boundary conditions. Thus, as in the preceding case, the function $\mathbf{p}_I$ should be set as a double-valued function.

The solutions found here for the kinematic optimum spatial turn problem determine only the necessary conditions for optimization. This means that all obtained solutions must be analyzed to choose the one of them having the minimum value of the criterion (turn time).

§ 6.3. OPTIMUM CONTROL FOR A BOUNDED MAGNITUDE OF THE ANGULAR VELOCITY VECTOR

We will consider the case of control when the boundary of the set of permissible controls is a sphere, i.e., when there is a limit on the magnitude of the angular velocity. We will consider the first form of the kinematic equation. In this case, the set Ω_E (6.2) will be determined by the condition:

$$\omega_1^2 + \omega_2^2 + \omega_3^2 \leqslant \omega_0^2. \tag{6.27}$$

where ω_0 is the magnitude of the maximum allowable magnitude of the angular velocity. Such is, evidently, the unique control, for which the set Ξ_j (6.5) is the same for the second form of the kinematic equation, i.e., / 251

$$\Omega_1^2 + \Omega_2^2 + \Omega_3^2 \leqslant \omega_0^2. \tag{6.28}$$

The condition for the maximum (6.9), with consideration of the representation of the Hamiltonian in the form (6.16), reduces to the requirement that the scalar product of the control vector ω_E and the conjugate vector $\mathbf{p}_E$ be maximum, i.e., obviously, the following equality must be satisfied:

$$\omega_E = h p_E,$$

where h is a proportionality constant. In addition, the value of ω_E must be chosen as the maximum possible, i.e.:

$$\omega_E = \omega_0.$$

If it is required that the length of the conjugate vector $\mathbf{p}_E$ be one (i.e., the length of $\mathbf{c}_E$ must also equal one in this case), then the two conditions written above for the maximum are written in the form

$$\omega_E = \omega_0 p_E. \tag{6.29}$$

The conjugate control vector $\mathbf{p}_E$ is determined, in accordance with the solution of this problem, by the expression (6.20) and, therefore, the optimum control (6.29) will be determined by the following relation:

$$\omega_E = \omega_0 \tilde{\Lambda} \circ c_E \circ \Lambda. \tag{6.30}$$

It should be noted that the obtained expression is the optimum control specified directly as a function of the phase coordinates, i.e., we have the solution of the optimum control synthesis problem. The magnitude of the vector $\mathbf{c}_E$, which obviously equals the initial value of the conjugate control vector:

$$c_E = p_E(t_0),$$

must be chosen such that the boundary conditions of the motion (6.3) are satisfied.

In order to obtain the expression for $\mathbf{c_E}$ through the boundary conditions of the quaternion $\Lambda(t)$, we substitute the solution (6.30) into the quaternion equation of motion (6.1) and have the / 252 following:

$$2\dot{\Lambda} = \Lambda \circ \omega_E = \Lambda \circ (\omega_0 \tilde{\Lambda} \circ c_E \circ \Lambda) = \omega_0 c_E \circ \Lambda.$$

The obtained equation coincides in form with the kinematic Equation (3.12) for a constant angular velocity vector equaling $\omega_0 c_E$. The solution of the equation is the function [cf., the solution (3.18)]:

$$\Lambda(t) = e^{\frac{1}{2}\omega_0 c_E t} \circ \Lambda_0, \tag{6.31}$$

satisfying the initial condition. In order that this function satisfy the final condition, it is necessary that the following condition be satisfied (at the time T):

$$\Lambda_T = e^{\frac{1}{2}\omega_0 c_E T} \circ \Lambda_0.$$

We find from this condition that

$$e^{\frac{1}{2}\omega_0 c_E T} = \Lambda_T \circ \tilde{\Lambda}_0. \tag{6.32}$$

Since $\mathbf{c_E}$ is a unit vector, then,

$$e^{\frac{1}{2}\omega_0 c_E T} = \cos\left(\frac{1}{2}\omega_0 T\right) + c_E \sin\left(\frac{1}{2}\omega_0 T\right).$$

Hence, we obtain the following expression for the vector $\mathbf{c_E}$ (except for the case when $\omega_0 T = 2\pi$, which corresponds to the identity transformation)

$$c_E = \frac{1}{\sin\frac{1}{2}\omega_0 T}\left(\Lambda_T \circ \tilde{\Lambda}_0 - \cos\frac{1}{2}\omega_0 T\right). \tag{6.33}$$

By substituting this value into the solution (6.31), we find

$$\Lambda(t)=$$
$$=\left[\cos\frac{1}{2}\omega_0 t+\frac{1}{\sin\frac{1}{2}\omega_0 T}\left(\Lambda_\tau\circ\tilde{\Lambda}_0-\cos\frac{1}{2}\omega_0 T\right)\sin\frac{1}{2}\omega_0 t\right]\circ\Lambda_0= \tag{6.34}$$
$$=\Lambda_0\left(\cos\frac{1}{2}\omega_0 t-\sin\frac{1}{2}\omega_0 t\operatorname{ctg}\frac{1}{2}\omega_0 T\right)+\Lambda_\tau\frac{\sin\frac{1}{2}\omega_0 t}{\sin\frac{1}{2}\omega_0 T}.$$

i.e., we obtain the solution of the kinematic optimum turn problem for this case of limitations on the control as a function of the boundary conditions directly. / 253

It is of interest to represent the optimum control function (6.30) with the help of the relations (6.32) and (6.33) for this case:

$$\omega_E(t)=\frac{\omega_0}{\sin\frac{1}{2}\omega_0 T}\left(\tilde{\Lambda}\circ\Lambda_\tau\circ\tilde{\Lambda}_0\circ\Lambda-\cos\frac{1}{2}\omega_0 T\right).$$

The obtained result does not depend on which moment of time the variable quaternion is taken, since we have, in accordance with (6.31) and (6.32):

$$\tilde{\Lambda}\circ\Lambda_\tau\circ\tilde{\Lambda}_0\circ\Lambda=\tilde{\Lambda}_0\circ e^{-\frac{1}{2}\omega_0 \varepsilon E t}\circ e^{\frac{1}{2}\omega_0 \varepsilon E T}\circ e^{\frac{1}{2}\omega_0 \varepsilon E t}\circ\Lambda_0=\tilde{\Lambda}_0\circ\Lambda_\tau.$$

Hence, we obtain finally the expression for optimum control:

$$\omega_E(t)=\frac{\omega_0}{\sin\frac{1}{2}\omega_0 T}\left(\tilde{\Lambda}_0\circ\Lambda_\tau-\cos\frac{1}{2}\omega_0 T\right). \tag{6.35}$$

Thus, the optimum trajectory in the case of control under consideration is a plane turn (6.34), which is performed with the constant maximum possible angular velocity (6.35) The direction of this velocity is determined by the final and initial positions of the body. Hence, it follows that a plane Euler turn of the rigid body with the maximum allowable velocity

is the optimum turn with respect to time for the case when the control is a function of the angular velocity, which is limited in magnitude.

The remark was made in the preceding paragraph that the quantity $\mathbf{c_E}$ (and $\mathbf{p_E}$) should be taken as double-valued. We will explain the meaning of this requirement. According to (6.30), the two signs of $\mathbf{c_E}$ (independently of the sign of the quaternion $\mathbf{\Lambda}$) will correspond to the different signs of ω_ε and, consequently, to the different directions of rotation. Both these solutions satisfy the optimization conditions and give a turn respectively by the angle $\omega_0(T-t_0)=\vartheta$ or $\omega_0(T-t_0)=2\pi-\vartheta$, where ϑ is the turn angle.

Of these two solutions, only one satisfies the condition / 254
for the minimum turn time (except for a turn by the angle π). Accordingly, the following sufficient condition for optimization must be satisfied for the minimum turn time:

$$\omega_0(T-t_0)\leqslant\pi. \tag{6.36}$$

One can see further that the quaternion $\mathbf{\Lambda}_T\circ\tilde{\mathbf{\Lambda}}_0$ defining the control is just the mismatch quaternion (for the two given positions of the body) in the quaternion control problems considered in Chapter 5. Hence, it follows that control in the kinematic problem will be optimum, if the correction velocity is generated as:

$$\Omega_{ki}=\Omega_{\max}\frac{\lambda_i}{\sqrt{1-\lambda_0^2}}\operatorname{sign}\lambda_0 \qquad (i=1,2,3), \tag{6.37}$$

where $\Omega_{\max}$ is the maximum value of the magnitude of the correction angular velocity. The control (5.6), for example, is close to such control.

In conclusion, we consider the solution of this problem for the second form of the kinematic equation. It was shown above that the limitation on the allowable control (6.28) coincides with the limitation (6.27). Hence, it follows that the condition for the maximum (6.12), with consideration of the Hamiltonian (6.24), is written in the form of the two conditions:

$$\omega_I = hp_I, \quad \omega_I = \omega_0,$$

analogous to the conditions for the first form of the kinematic equation.

Let us require that the magnitude of the vector $\mathbf{p}_I$ be equal to one. Then both of these conditions can be written

$$\omega_I = \omega_0 p_I, \tag{6.38}$$

analogous to the condition (6.29). By taking into account that the conjugate control vector $\mathbf{p}_I$ is determined by Expression (6.26) in accordance with the solution of the optimum problem, we obtain for optimum control (6.38) the following relations

$$\omega_I = \omega_0 \Lambda \circ c_I \circ \tilde{\Lambda}, \tag{6.39}$$

Obviously, the vector $\mathbf{c}_I$ is determined as the initial value of the conjugate control vector: / 255

$$c_I = p_I(t_0).$$

To obtain the optimum trajectory of the motion, we substitute the optimum control (6.39) into the quaternion equation of motion (6.4):

$$2\dot{\Lambda} = \omega_I \circ \Lambda = \omega_0 \Lambda \circ c_I.$$

This equation is the kinematic equation [of the type (3.11)] for a constant angular velocity vector $\omega_0 c_I$. Its solution is the following function:

$$\Lambda(t) = \Lambda_0 \circ e^{\frac{1}{2}\omega_0 c_I t}. \qquad (6.40)$$

The solution (6.40) satisfies the initial condition (6.3). In order that this solution pass through the final point Λ_T, it is necessary to choose the vector $\mathbf{c}_I$ in a corresponding manner. By equating the right part of (6.40) at time T to the final condition (6.3), we obtain

$$\Lambda_0 \circ e^{\frac{1}{2}\omega_0 c_I t} = \Lambda_T,$$

from which we find, by repeating all the calculations analogous to the preceding case, that the vector $\mathbf{c}_I$ must be determined by the relation $(\omega_0 T \neq 2\pi)$:

$$c_I = \frac{1}{\sin\frac{1}{2}\omega_0 T}\left[\tilde{\Lambda}_0 \circ \Lambda_T - \cos\frac{1}{2}\omega_0 T\right]. \qquad (6.41)$$

By substituting this value into Equation (6.40), we obtain exactly the same trajectory defined by relation (6.34) as in the preceding case. It is natural that the same results are obtained, since the limitation (6.27) and, consequently, the optimum motion must be the same, although they can have different formal descriptions in different coordinate systems.

Let us find the optimum control function $\omega_I(t)$. By substituting the value of the vector $\mathbf{c}_I$ from (6.41) into the equality (6.39), we obtain / 256

$$\omega_I(t) = \frac{\omega_0}{\sin\frac{1}{2}\omega_0 T}\left[\Lambda \circ \tilde{\Lambda}_0 \circ \Lambda_T \circ \tilde{\Lambda} - \cos\frac{1}{2}\omega_0 T\right].$$

One can show from the solution (6.40) that:

$$\Lambda \circ \tilde{\Lambda}_0 \circ \Lambda_T \circ \tilde{\Lambda} =$$
$$= \Lambda_0 \circ e^{\frac{1}{2}\omega_0 c_I t} \circ \tilde{\Lambda}_0 \circ \Lambda_0 \circ e^{\frac{1}{2}\omega_0 c_I T} \circ e^{-\frac{1}{2}\omega_0 c_I t} \circ \tilde{\Lambda}_0 = \Lambda_T \circ \tilde{\Lambda}_0.$$

Hence, we obtain the final expression for the optimum control

$$\omega_I(t) = \frac{\omega_0}{\sin \frac{1}{2}\omega_0 T}\left[\Lambda_T \circ \tilde{\Lambda}_0 - \cos\tfrac{1}{2}\omega_0 T\right]. \qquad (6.42)$$

Obviously, the control (6.42) defines the same plane Euler turn as in the preceding case.

The fact that the mapping quaternions ω_I (6.42) and ω_E (6.35) determine the same angular velocity vector can also serve as proof of this. Actually, one can verify that the relation:

$$\omega_E = \tilde{\Lambda} \circ \omega_I \circ \Lambda,$$

is satisfied, which relates the two mappings of one vector. For this, it is sufficient to return to the calculations carried out above, from which it follows that:

$$\Lambda \circ \tilde{\Lambda}_0 \circ \Lambda_T \circ \tilde{\Lambda} = \Lambda_T \circ \tilde{\Lambda}_0, \quad \tilde{\Lambda} \circ \Lambda_T \circ \tilde{\Lambda}_0 \circ \Lambda = \tilde{\Lambda}_0 \circ \Lambda_T$$

and, consequently, according to (6.42) and (6.35), the relations presented above are valid.

§ 6.4. OPTIMUM CONTROL WITH LIMITATIONS ON THE COMPONENTS OF THE ANGULAR VELOCITY VECTOR

The following case of control occurs when each component, the projections of the angular velocity vector, are limited. There is practical value in having an analysis of the control, when the limitations on the projections on the angular velocity vector/ 257 are specified in the body basis **E**. The set Ξ_E (6.2) will be determined by the conditions

$$|\omega_{Ei}| \leqslant \Omega_i \quad (i = 1, 2, 3), \qquad (6.43)$$

where Ω_i are the magnitudes of the maximum allowable components of the angular velocity vector along the body axes $\mathbf{e}_i$. In the

space ω_{Ei}, the set Ξ_E will have the form of a parallelapiped whose sides equal Ω_i.

The condition for the maximum of the Hamiltonian (6.9) or (6.16) with such a region of allowable controls reduces to the requirement of maximizing the quantity $\omega_E \cdot p_E$, which occurs when each component of the vector ω_E acquires the maximum value in the direction of the vector $\mathbf{p}_E$:

$$\omega_{Ei} = \Omega_i \operatorname{sign}(p_{Ei}) \quad (i = 1, 2, 3). \tag{6.44}$$

We denote the vectors running from the origin of the coordinate system to the apices of the parallelapiped Ξ_E (i.e., along its diagonals) by ω_m $(m = 1, 2, \ldots, 8)$, without specifying at present an order for assigning the indices. Each of these vectors is obtained when all three components of ω_E take their extremum values (6.42).

We denote the vectors running from the origin of the coordinate system perpendicular to each edge of the parallelapiped Ξ_E by $\omega_{\gamma l}$. One can see that there are twelve such vectors in all, i.e., $l = 1, 2, \ldots, 12$. The vectors $\omega_{\gamma l}$ are the values of the vector ω_E when two of its components take extremum values, and one vanishes.

Finally, we denote the vectors running from the origin of the coordinate system perpendicular to each face of the parallelapiped Ξ_E by ω_{An}. The magnitude of each of these vectors equals Ω_i. We have six such vectors in all, i.e., n = 1, 2, ..., 6. Each of the vectors equals the vector ω_E, for which two components have vanished, and one takes the extremum value.

Let us now consider the conditions (6.44) for optimizing control and determine which motions these conditions exclude. When the vector $\mathbf{p}_E(t)$ is located anywhere outside the coordinate / 258 planes, i.e., for $p_{Ei} \neq 0$ (i = 1, 2, 3), the conditions (6.44) determine the motion which occurs with one of the maximum angular velocity vectors ω_m. If the vector $\mathbf{p}_E(t)$ falls on a coordinate pl ·e, i.e., one (or two) of its components vanishes, then this case is singular control.

Analysis of singular controls has shown that it is necessary to postulate the following hypotheses (which does not follow from the principle of the maximum):

1. If any component of p_{Ei} vanishes, then the corresponding component of the velocity ω_i can take three values: $+\Omega_i$, 0, $-\Omega_i$. In other words, the function sign (p_{Ei}) in the expression (6.44) can take the values +1, 0, -1 at zero.

In I, Paragraph 6.7, it was shown that such a postulate is very significant, since it permits one to extend the number of allowable trajectories of the motion to the necessary number.

Extension of the possible motions, which satisfy the necessary conditions of optimization, occurs thanks to this property of the functions (6.44): firstly, motions with one of the angular velocities $\omega_{\varepsilon l}$ become possible; secondly, motions with the velocities $\omega_{\varepsilon n}$ are possible. The first case occurs, when the vector $\mathbf{p}_E$ falls on a coordinate plane and one of the component p_{Ei} vanishes. According to the assumed hypothesis, the component ω_{Ei} can equal $+\Omega_i$, 0, $-\Omega_i$, i.e., motion with the angular velocity $\omega_{\varepsilon l}$ is admissible. In the second case, the vector $\mathbf{p}_E$ falls on the coordinate axis, i.e., two of its components become zero, in connection with which motion with the

velocity $\omega_{\mathcal{A}n}$ is possible (in I, Paragraph 6.7, it will be shown that other cases of motion with the velocity $\omega_{\mathcal{A}n}$ are possible). Thus, we have $m + n + l = 26$ admissible angular velocity vectors in all, and rotation with each of them satisfies the optimization condition.

We denote the whole set of these velocities by ω_k ($k = 1, 2, \ldots, 26$), and will call them the extremum angular velocities. According to (6.44), all trajectories of the motion must consist / 259 of segments on each of which rotation with these angular velocities occurs. Rotation on each segment is performed with a constant angular velocity; the solution of the kinematic Equation (3.79) is known for such motion. This fact permits one to completely solve this optimum kinematic turn problem.

As a result of solving the problem, answers should be given for such questions as: of how many segments of the motion does the entire trajectory consist; what sequence of rotation with the extremum angular velocities is required to accomplish the spatial turn; by how much does the optimum control for this case differ from the control (6.37) considered in the preceding paragraph.

Before turning to an investigation of the motion with the control (6.44), we will make some general assumptions.

2. The required spatial turn determined by the boundary conditions (6.3) specify one initial value of the conjugate control vector $\mathbf{p}_E$ as minimum.*

*It will be shown later that the boundary condition can determine not one, but several different motions.

3. The vector $\mathbf{p}_E$ will be considered double-valued; we will assume that the boundary conditions of the motion (6.3) specify two values of the vector $\mathbf{p}_E$ simultaneously. The motion is then executed by one of the two values of the vector $\mathbf{p}_E$ which is maintained throughout the subsequent motion.*

Qualitative investigation of the motion. We recall that the relations (6.44), (6.20), and (6.1), together with the boundary conditions (6.3), form a closed system of equations describing the optimum kinematic turn. According to (6.44), all motion will consist of segments of rotations with velocities ω_k. Thus, the angular velocity of the optimum turn will be a piece-wise constant function. The discontinuities of the angular velocity / 260
function will occur when the coordinate planes are intersected by the vector $\mathbf{p}_E$.

Let us consider how the relation (6.20) is satisfied for trajectories of such a nature. We denote the final value of the quaternion of the motion at the $k^{\underline{th}}$ segment by Λ_k. This value will then be the initial value of the quaternion in the $k+1)^{\underline{st}}$ of the motion, which occurs with the velocity ω_{k+1}. The solution of the kinematic equation of motion on the $(k+1)^{\underline{st}}$ segment will have the form

$$\Lambda(t'_{k+1}) = \Lambda_k \circ e^{\frac{1}{2}\bullet_{k+1} t'_{k+1}}, \tag{6.45}$$

where $t_k \leqslant t'_{k+1} \leqslant t_{k+1}$, t'_{k+1} is the time variable on the $(k+1)^{\underline{st}}$ segment of the motion. The value $\Lambda(t_{k+1})$ will equal Λ_{k+1}.

We have, by analogy to (6.45), for the first segment of the motion:

*The hypothesis 3 is not as clear as the hypothesis 2; its justification will be presented in I, Paragraph 6.7.

$$\Lambda_1 = \Lambda_0 \circ e^{\frac{1}{2}\omega_1 t_1},$$

where Λ_0 is determined by the initial condition (6.3). If the motion begins at the first segment and ends at the $k^{\underline{\text{th}}}$ segment, then we obtain the following value for the final quaternion, according to (6.45):

$$\Lambda_\tau = \Lambda_0 \circ e^{\frac{1}{2}\omega_1 t_1} \circ \ldots \circ e^{\frac{1}{2}\omega_k t_k},$$

which corresponds to theorem .4.

Let us have the initial value of the conjugate control vector $\mathbf{p}_{E0} = \mathbf{c}_E$. At the end of the first segment of the motion, it will be determined, according to the relation (6.20), by the expression

$$p_{E1} = \tilde{\Lambda}_1 \circ c_E \circ \Lambda_1 = e^{-\frac{1}{2}\omega_1 t_1} \circ \tilde{\Lambda}_0 \circ c_E \circ \Lambda_0 \circ e^{\frac{1}{2}\omega_1 t_1}.$$

We replace the quantity $\tilde{\Lambda}_0 \circ c_E \circ \Lambda_0$ by $\mathbf{p}_{E0}$, then

$$p_{E1} = e^{-\frac{1}{2}\omega_1 t_1} \circ p_{E0} \circ e^{\frac{1}{2}\omega_1 t_1}.$$

This final value for the first segment will be the second value of the second segment of the motion and, consequently, we will have at the end of the second segment of the motion: / 261

$$p_{E2} = e^{-\frac{1}{2}\omega_2 t_2} \circ p_{E1} \circ e^{\frac{1}{2}\omega_2 t_2}.$$

With the continuous application of this formula, we obtain the following value of the vector $\mathbf{p}_E$ at the end of the motion:

$$p_{E\tau} = e^{-\frac{1}{2}\omega_k t_k} \circ \ldots \circ e^{-\frac{1}{2}\omega_1 t_1} \circ p_{E0} \circ e^{\frac{1}{2}\omega_1 t_1} \circ \ldots \circ e^{\frac{1}{2}\omega_k t_k} = \tilde{\Lambda}_\tau \circ c_E \circ \Lambda_\tau.$$

Hence, it can be seen that the relation (6.20) also remains valid for piece-wise-continuous angular velocity functions. The quantity

$$e^{\frac{1}{2}\bullet_1 t_1}\circ\ldots\circ e^{\frac{1}{2}\bullet_k t_k}$$

is the resultant motion. One can see that it is uniquely determined by the boundary conditions (6.3). We denote it in the following manner:

$$\Lambda_\Sigma = e^{\frac{1}{2}\bullet_1 t_1}\circ\ldots\circ e^{\frac{1}{2}\bullet_k t_k} = \tilde{\Lambda}_0\circ\Lambda_T. \tag{6.46}$$

This resultant quaternion also determines the relation between the initial and final values of the conjugate control vectors:

$$p_{\mathcal{E}T} = \tilde{\Lambda}_\Sigma\circ p_{\mathcal{E}0}\circ\Lambda_\Sigma. \tag{6.47}$$

According to the hypothesis 2, the quantity Λ_Σ must determine the initial value of the vector $\mathbf{p}_{E0}$. Having carried out the quantitative value of the problem, we will turn the inverse route: we will assume that the vector $\mathbf{p}_{E0}$ is given and analyze the obtained motion and the resultant quaternion.

Let the quantities Ω_i (6.43) be equal, i.e., the region of allowable controls $\Xi_\mathcal{E}$ is a cube with sides $2\Omega_i$. Without loss of generality, we can set $\Omega_i = 1$, then the cube $\Xi_\mathcal{E}$ can be circumscribed by a sphere of radius $\sqrt{3}$ (Figure 6.1). In accordance with the relations (6.44) and hypothesis 1 relating to the function sign (9), the motion will take place with angular velocities/ 263 equaling: a) the half-diagonals of the cube ω_m, the magnitude of whose angular velocity equals $\sqrt{3}$; b) the half-diagonals of the squares (the faces of the cube) $\omega_{\gamma i}$, the magnitude of whose

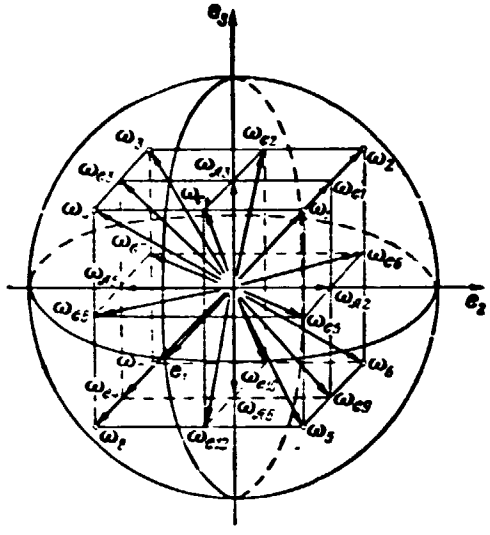

Figure 6.1.

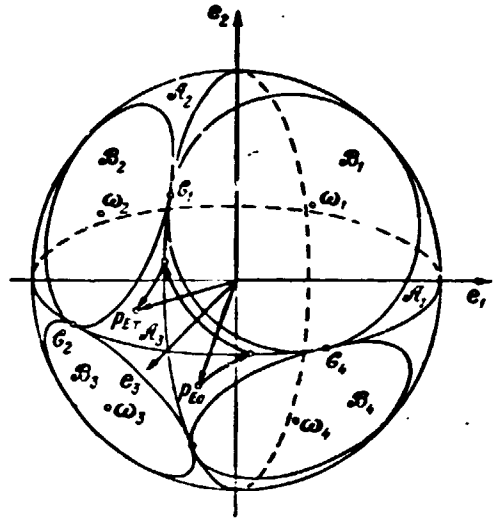

Figure 6.2.

angular velocity equals $\sqrt{2}$; c) the half-edges of the cube ω_{An}, the magnitude of whose angular velocity equals 1.

We introduce the enumeration of the velocities as is shown in Figure 6.1 (it is true, not for all ω_k). We will assume that the initial value of the vector $\mathbf{p}_{E0}$ is given and consider the various cases of motion, which are determined by its position. We will present the results of this analysis.

It is seen that, from Figure 6.2, there are eight regions $\mathcal{B}_m$ $(m = 1, 2, \ldots, 8)$ on the sphere. If the vector $\mathbf{p}_{E0}$ is located in such a region, then the motion is performed with the velocity ω_m and, as a result of the motion, the vector $\mathbf{p}_E$ does not change the sign of its components. Actually, with rotation about a cone in any direction, the vector $\mathbf{p}_E$, being located in the region $\mathcal{B}_m$, does not intersect or touch the coordinate planes. The boundary of this motion will be the position of the vector $\mathbf{p}_{E0}$ for which the tangency to the coordinate plane occurs and, since the region of allowable controls is a cube, the tangency

to three planes occurs simultaneously. Such a boundary will be called the separatrix, and we will consider it as not belonging to the region $\mathscr{B}_m$.

With any position of the vector $\mathbf{p}_{E0}$ inside the region $\mathscr{B}_m$, only one case of motion is executed, when the resultant rotation has one segment of motion with the velocity ω_m. The corresponding quaternion will be denoted by Λ_m. The resultant quaternion in this case will be determined by only one component:

$$\Lambda_m = e^{\frac{1}{2}\bullet m t}. \tag{6.48}$$

The set of points for the location of the end of the vector $\mathbf{p}_{E0}$ on the sphere and in the region $\mathscr{B}_m$ corresponds to the same motion as when the vectors $\mathbf{p}_{E0}$ and ω_m coincide. Hence, it follows that region $\mathscr{B}_m$ is equivalent to the one point determined by the intersection of the vector ω_m with the sphere. The distinguishing feature of this point is that, when passing through it, the vector $\mathbf{p}_E(t)$ does not change its position as a result of / 264 the motion determined by it (the vector remains constant). Such points will be called stationary.

One can further see that there are six regions $\mathscr{A}_n$ (n = 1, 2, ..., 6) on the sphere, distinct from the regions $\mathscr{B}_m$ and separated from them by the separatrix (we will consider the separatrix as also not belonging to the region $\mathscr{A}_n$). When located in one of the regions ... the vector $\mathbf{p}_E$ can reach and intersect the coordinate planes; it is in these regions that motion with different angular velocities are possible. Then the rotational velocities are ω_m, and the sequence of velocity shifts and the length of each segment of the motion with constant velocity are determined by the position of the vector $\mathbf{p}_{E0}$. The types of motions, which arise when the vector $\mathbf{p}_{E0}$ is located in the region $\mathscr{A}_n$, will be investigated further.

When the vector $\mathbf{p}_{E0}$ is located on the separatrix, singular cases of motion occur, which arise when the vector $\mathbf{p}_E$ falls at the point $\mathscr{C}_l$ (cf., Figure 6.2.). One component of this vector then vanishes. In accordance with hypothesis 1, the value of the function sign (0) at these points can equal +1, 0, -1. In those cases when the value of sign (0) equals ±1, the rotational angular velocities will equal one of the values ω_m according to (6.44), and the vector $\mathbf{p}_E$ leaves the point $\mathscr{C}_l$ as a result of the rotation with this velocity. However, if the value of the function sign (0) equals zero, then a segment of motion with the velocity $\omega_{\gamma l}$ is possible. For such motion, the vector $\mathbf{p}_E$ (passing through the point $\mathscr{C}_l$) coincides with the vector $\omega_{\gamma l}$ and, consequently, does not change its position as a result of the motion.

Thus, we have another case of a stationary point: when the vector $\mathbf{p}_{E0}$ passes through the point $\mathscr{C}_l$ and the value of the function sign (0) = 0, the motion consists of one segment of rotation with the velocity $\omega_{\gamma l}$. We denote the corresponding quaternion by $\Lambda_{\gamma l}$:

$$\Lambda_{\gamma l}=e^{\frac{1}{2}\cdot\gamma l^{\prime}}, \tag{6.49}$$

There is another type of stationary point, at which motion can be executed, which consists of one segment. This point is located at the center of the region $\mathscr{A}_n$ at the point of intersection of the sphere with the coordinate axis. If the vector $\mathbf{p}_E$ passes through this point, then two of its components can vanish. In accordance with hypothesis 1, the control (6.44) then permits rotation with the velocity $\omega_{\mathscr{A}n}$. Such rotation does not change the vector $\mathbf{p}_E$, i.e., it is also stationary. We denote the quaternion for rotation with the velocity $\omega_{\mathscr{A}n}$ by $\Lambda_{\mathscr{A}n}$:

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$$\Lambda_{\mathcal{A}n} = e^{\frac{1}{2}\omega_{\mathcal{A}n} t}. \qquad (6.50)$$

Thus, we have shown that there are positions of the vector $\mathbf{p}_{E0}$ on the sphere, which determine motion with any of the extremum velocities ω_k. When the vector $\mathbf{p}_{E0}$ is located outside the point in the regions $\mathcal{A}_n$ and on the separatrix, motion is executed which has several segments of rotation with the extremum velocities.

We will classify the possible motions, depending on the number of segments and on the sequence of rotational angular velocities.

We will first note the considered cases of stationary rotation, when the resultant motion consists of one segment.

We have the following types of motion (trajectories):

I. $\Lambda_\Sigma = \Lambda_m$ [when the vector $\mathbf{p}_{E0}$ is located in the region $\mathcal{B}_m$, cf., (6.48)];

II. $\Lambda_\Sigma = \Lambda_{\mathcal{A}n}$ [when the vector $\mathbf{p}_{E0}$ is located at the points of intersection of the coordinate axes with the sphere, cf., (6.50)];

III. $\Lambda_\Sigma = \Lambda_{\mathcal{B}l}$ [when the vector $\mathbf{p}_{E0}$ is located at the points $\mathcal{B}_l$, cf., (6.49)].

We will now consider the case when the vector $\mathbf{p}_{E0}$ is located in the region $\mathcal{A}_n$. To be definite (without loss of generality), we will assume that the vector $\mathbf{p}_{E0}$ is located in the region $\mathcal{A}_3$ in the first quadrant, which is formed by the positive directions

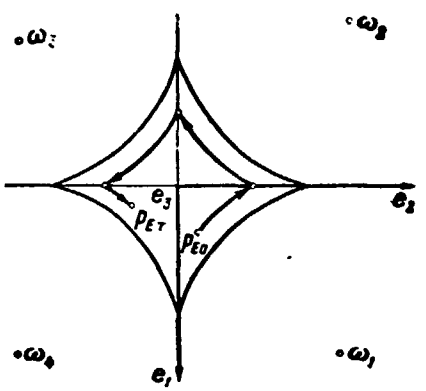

Figure 6.3.

of the axes e_1, e_2, e_3. The obtained trajectories of the vector $\mathbf{p}_E$ are presented in Figure 6.3. We note/ 266 that, when the vector $\mathbf{p}_{E0}$ is located anywhere in the region $\mathcal{A}_n$, except the center of this region, the motion occurs only with the velocities ω_m. Actually, even when the vector $\mathbf{p}_E$ falls on the coordinate plane, when one of its components vanishes, the vector $\mathbf{p}_E$ does not remain in this plane, but intersects it.

We will consider the passage of the vector $\mathbf{p}_E$ through the plane e_2e_3 as an example (cf., Figure 6.3). The motion occurs in quadrant $e_1e_2e_3$ with the velocity ω_1, as a result of which the vector $\mathbf{p}_E$ falls on the plane e_2e_3 and the component $\mathbf{p}_{E1}$ vanishes. According to hypothesis 1, the function sign (0) takes the values +1 or 0, or -1, and the motion in these cases will occur with velocities ω_1, ω_{c1} or ω_2, respectively. In all cases, the vector $\mathbf{p}_E$ must intersect the plane e_2e_3. According to Figure 6.3, we have the following types of motions:

IV $\Lambda_\Sigma = \Lambda_1 \circ \Lambda_2$;
V $\Lambda_\Sigma = \Lambda_1 \circ \Lambda_2 \circ \Lambda_3$;
VI $\Lambda_\Sigma = \Lambda_1 \circ \Lambda_2 \circ \Lambda_3 \circ \Lambda_4$;
VII $\Lambda_\Sigma = \Lambda_1 \circ \Lambda_2 \circ \Lambda_3 \circ \Lambda_4 \circ \Lambda_1'$.

The case, when the end of the arc Λ_1' (i.e., the vector $p_{E\tau}$) falls at the point $\mathbf{p}_{E0}$, is possible. We will call such a case a closed cycle. Obviously, a further increase in the number of / 267 segments is possible when the trajectory becomes longer than one closed cycle and forms a double cycle, etc. (the number of

segments can increase indefinitely). In view of the fact that any sequence of rotations must correspond to the given rotation Λ_Σ, an increase in the number of segments is possible, in principle, only when each of them is decreased, which occurs as the vector $\mathbf{p}_{E0}$ approaches the center of the region $\mathcal{A}_3$. If the vector $\mathbf{p}_{E0}$ is located close to the center, then the motion will consist of an infinitely large number of infinitesimal rotations, i.e., we will have the so-called sliding mode. The resultant velocity of such motion will equal $\omega_{\mathcal{A}3}$.

One can see that the obtained case of stationary rotation (6.50) is a limit of an infinite number of cyclic motions tending toward the trajectory of the type II. Hence, it also follows that the rotation (6.50) is isolated and cannot be joined with other rotations,* in contrast to the rotations (6.48) and (6.49).

We will show that the number of arbitrary parameters for any of the trajectories of motion determined above does not exceed three, i.e., the number necessary for specifying an arbitrary rigid body turn. Obviously, trajectories consisting of m segments have m free parameters (rotation angles). Hence, it follows that trajectories consisting of one, two, and three segments of motion are uniquely determined by the resultant quaternion Λ_Σ. Nonetheless, the conditions for the maximum (i.e., the conditions of the continuous motion of the vector $\mathbf{p}_E$) impose limitations on the magnitudes of the rotation angles. One can see that the following inequalities must be satisfied on the trajectory of the type V:

$$\vartheta_2 \geqslant \vartheta_1, \quad \vartheta_2 \geqslant \vartheta_3, \quad \vartheta_2 < \frac{2\pi}{3}. \tag{6.51}$$

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*In I, paragraph 6.7, another hypothesis will be considered, for which motion of the type II can be converted into other types of motions.

Hence, it follows that only the trajectory having the most prolonged middle segment of the motion can be optimum. If the conditions (6.51) are satisfied, then the trajectory of the motion of the vector $\mathbf{p}_E$ can be associated with any resolution of the motion Λ_Σ into three rotations Λ_m. We have, for the trajectory of type VI, the length of the second segment equaling the length of the third, and the number of free parameters also equals three: / 268

$$\vartheta_2 = \vartheta_3 < \frac{2\pi}{3}, \quad \vartheta_1 \leqslant \vartheta_2 = \vartheta_3, \quad \vartheta_4 \leqslant \vartheta_2 = \vartheta_3. \tag{6.52}$$

An analogous case occurs for five and more segments of motion.

We will finally consider the types of motion which arise when the vector $\mathbf{p}_{E0}$ is located on the separatrix (Figure 6.4). Motions are possible in this case, which consists of one segment, i.e., the trajectories of type I and III discussed previously. One can see that, if the vector $\mathbf{p}_E$ passes through the point $\mathscr{C}_1$ and the function sign (0) does not take its zero value, then the trajectory of the type IV can be executed. However, in contrast to the preceding case, segments of motion with a rotation angle exceeding $\frac{2\pi}{3}$ are possible in this trajectory. For this reason, we did not impose limitations on the rotation angles for trajectories of the type IV. With subsequent motion, the vector $\mathbf{p}_E$, under these same conditions, can execute the trajectory of type V with the angle $\vartheta_2 = \frac{2\pi}{3}$ and the following types of trajectory. However, as will be shown later, trajectories which contain more than two segments of rotation with a rotation angle $\frac{2\pi}{3}$, give a resultant rotation angle greater than π. In this connection, the sign of the inequality occurs in the first relation (6.52). / 269

The following trajectory will be a new type of trajectory for the case when the vector $\mathbf{p}_{E0}$ is found on the separatrix (see Fig. 6.4):

VIII $\Lambda_\Sigma = \Lambda_1 \circ \Lambda_{\mathscr{C}_1}$.

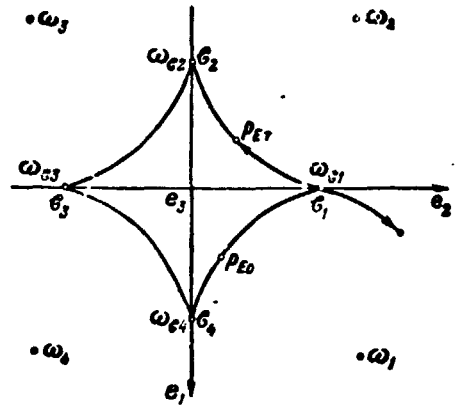

Figure 6.4.

This trajectory arises when the vector $\mathbf{p_E}$ calls at the point $\mathscr{C}_1$ and the function sign (0) takes the value zc . Let the vector $\mathbf{p_{E0}}$ be located at the point $\mathscr{C}_1$ and the rotation (6.49) correspond to it. However, if it is allowed that with a lapse of some time the function sign (0) changes its value from 0 to +1, then this will correspond to the trajectory:

$$\text{IX } \Lambda_\Sigma = \Lambda_{\sigma 1} \circ \Lambda_1.$$

The following trajectory will be of this type:

$$\text{X } \Lambda_\Sigma = \Lambda_1 \circ \Lambda_{\sigma 1} \circ \Lambda_1'.$$

If the function sign (0) takes the value -1 after the zero value, then this case will correspond to the trajectory $\Lambda_\Sigma = \Lambda_{\sigma 1} \circ \Lambda_2$, which belongs to the type IX. However, if the motion begins with a rotation with the velocity ω_1, then another type of trajectory occurs:

$$\text{XI } \Lambda_\Sigma = \Lambda_1 \circ \Lambda_{\sigma 1} \circ \Lambda_2.$$

By reasoning analogously, one can obtain further such types of trajectories:

$$\text{XII } \Lambda_\Sigma = \Lambda_{\sigma 1} \circ \Lambda_2 \circ \Lambda_{\sigma 2},$$
$$\text{XIII } \Lambda_\Sigma = \Lambda_1 \circ \Lambda_{\sigma 1} \circ \Lambda_2 \circ \Lambda_{\sigma 2};$$
$$\text{XIV } \Lambda_\Sigma = \Lambda_1 \circ \Lambda_{\sigma 1} \circ \Lambda_2 \circ \Lambda_{\sigma 2} \circ \Lambda_3.$$

In these three cases, the rotation angle θ_2 (the quaternion Λ_2) equals $\frac{2\pi}{3}$. There cannot be a further increase of the segments of motion, since this would reduce to two rotations at a minimum by an angle $\frac{2\pi}{3}$, which gives in the sum the angle θ_Σ exceeding π. In II, Paragraph 6.7, it is shown that the trajectory of type XII with $\theta_2=\frac{2\pi}{3}$ is equivalent to the trajectories of type VII and IX, i.e., /270

$$\Lambda_{\gamma 1}\circ\Lambda_2\circ\Lambda_{\gamma 2}=\Lambda'_{\gamma 1}\circ\Lambda_2=\Lambda_2\circ\Lambda'_{\gamma 2}, \tag{6.53}$$

Addition of the arcs of the rotations $\Lambda_{\gamma 1}$ and $\Lambda_{\gamma 2}$ occurs on the trajectory of the type XII, and

$$\theta'_{\gamma 1}=\theta_{\gamma 1}+\theta_{\gamma 2}=\theta'_{\gamma 2}.$$

It follows from this property of the trajectory of the type XII, that the trajectory of type XIII is equivalent to the following trajectories:

$$\Lambda_\Sigma=\Lambda_1\circ\Lambda_{\gamma 1}\circ\Lambda_2\circ\Lambda_{\gamma 2}=\Lambda_1\circ\Lambda'_{\gamma 1}\circ\Lambda_2=\Lambda_1\circ\Lambda_2\circ\Lambda'_{\gamma 2}.$$

Obviously, we have, in the first case, the trajectory of type XI, and the second case determines a new type of trajectory, which we will also denote by the number XIII:

$$\text{XIII } \Lambda_\Sigma=\Lambda_1\circ\Lambda_2\circ\Lambda_{\gamma 2},\quad \theta_*=\frac{2\pi}{3}$$

(keeping in mind that for $\theta_2=\frac{2\pi}{3}$, this trajectory is equivalent to the trajectory of type XI). The following trajectory is also related to this same type of trajectory:

$$\text{XIII } \Lambda_\Sigma=\Lambda_{\gamma 1}\circ\Lambda_1\circ\Lambda_2,\quad \theta_1=\frac{2\pi}{3}.$$

We obtain, by analogy, that the trajectory of the type XIV reduces to:

XIV $\Lambda_\Sigma = \Lambda_1 \circ \Lambda_{\gamma 1} \circ \Lambda_2 \circ \Lambda_3 = \Lambda_1 \circ \Lambda_2 \circ \Lambda_{\gamma 2} \circ \Lambda_3$, $\theta_{\gamma 1} = \theta_{\gamma 2}$.

since $\theta_2 = \frac{2\pi}{3}$ always.

Thus, we have determined fourteen types of possible trajectories of the motion. One can see that the number of free parameters for each trajectory of the type VIII — XIV does not exceed three, i.e., the number of parameters which is uniquely (for given rotation directions) determined by the given resultant rotation Λ_Σ. Hence, it follows that all the component rotations for each trajectory type can be determined by the quantity Λ_Σ. / 271

On the other hand, if the resultant rotation (6.46) is specified, then several (of those enumerated) trajectories satisfying the optimization conditions can, generally speaking, correspond to it. This should be expected, proceeding, for example, from the fact that the optimization conditions are only necessary conditions; the conditions for selecting the shortest trajectory from all the given trajectories will be sufficient conditions for optimization.

In order to turn to an analysis of this question, it is necessary to establish how many of the fourteen types of trajectories enumerated anove can exist for a given resultant quaternion. Let the quaternion Λ_Σ be defined by the rotation angle θ_Σ and the vector ζ_Σ (1.11); the latter specifies a point on the sphere (the point of intersection of the vector ζ_Σ with the sphere). The angle θ_Σ and two parameters, the coordinates of the vector ζ_Σ, correspond to the three independent parameters (we will define them by the quantities γ_1, γ_2 and γ_3, the direction cosines of the vector ζ_Σ). We will consider the regions on the sphere, in which the ends of the vectors ζ_Σ will be located for each of the fourteen types of possible trajectories.

The trajectories of the types I, II, and III have one free parameter, i.e., the rotation angle, which must equal ϑ_Σ. For this reason, such trajectories have only one point on the sphere (i.e., the point is their region of existence). The trajectories of the type IV, VIII, IX, XII, which consist of two segments, have two free parameters and thus define on the sphere a line on which the ends of the vectors $\mathbf{\Lambda}_\Sigma$ are located. All the remaining trajectories have finite (in area) regions on the sphere. Of course, these regions are bounded by lines, which will be some boundary cases of the given motion. Each such boundary case arises with the transition from one type to another. The condition of such a transition is then a supplementary condition, which reduces the number of free parameters from three to two, which corresponds to a line on the sphere.

We will show that such boundary cases actually have two / 272
free parameters, i.e., they define a line on the sphere. Thus, for example, the region of the trajectory of the type V is bounded on one side by the cases when $\vartheta_1 = \vartheta_2$ or $\vartheta_2 = \vartheta_3$ (the number of free parameters then actually equals two: these cases determine the transition boundary between the trajectories of the type V and trajectories of the type VI. This region is bounded on the other side by the cases $\vartheta_1 = 0$ and $\vartheta_3 = 0$, i.e., by the trajectories of the type IV.

The region of the trajectory of type VI is bounded by the boundaries cited above with the trajectory of type V (which occurs when the conditions $\vartheta_1 = 0$ or $\vartheta_4 = 0$ are satisfied on the trajectory of type VI); this region is bounded on the other side by the cases $\vartheta_2 = \vartheta_3 = \vartheta_4$ or $\vartheta_1 = \vartheta_2 = \vartheta_3$ (these boundaries separate this type of trajectory from the trajectories of type VII).

The trajectory of type VII is separated from the latter type of trajectories by the boundaries: $\vartheta_2=\vartheta_3=\vartheta_4=\vartheta_1+\vartheta_1'$, $\vartheta_2=\vartheta_1=\vartheta_3=\vartheta_4$ and $\vartheta_2=\vartheta_3=\vartheta_4=\vartheta_1'$.

The trajectory of type X has only two boundaries: the first is determined by the condition $\vartheta_1=0$, when the boundary of the trajectory of type X is the trajectory of the type IX, and the second, by the condition $\vartheta_1'=0$, when the boundary is the trajectory of type VIII. The case $\vartheta_{\gamma 1}=0$ corresponds to the trajectory of type I, i.e., one point at which both boundaries, the trajectories of the type VIII and IX, converge (for $\vartheta_{\gamma 1}=0$).

The trajectory of type XI has the following boundaries: the trajectory of type IV for $\vartheta_{\gamma 1}=0$ and the trajectory of type VIII or XI for $\vartheta_1=0$ or $\vartheta_2=0$. When the angle ϑ_2 reaches the value $\frac{2\pi}{3}$, we obtain the boundary between the trajectories of type XI and the trajectories of type XIII (in the latter case, one should set $\vartheta_1=0$). Another type of boundary exists for $\vartheta_1=\frac{2\pi}{3}$, when we have, by analogy to (6.53):

$$\Lambda_1\circ\Lambda_{\gamma 1}=\Lambda_{\gamma 4}\circ\Lambda_1,$$

as a consequence of which, we obtain a second modification of the trajectory of type XIII:

$$\Lambda_\Sigma=\Lambda_{\gamma 4}\circ\Lambda_1\circ\Lambda_2.$$

The trajectories of type XIII and XIV e ist only for large / 273
angles ϑ_Σ and their existence regions are the following; the trajectory of type XIII has two free parameters, i.e., it is the line (boundary) separating the trajectory of type XI from the trajectory of type XIV. Thus, one can construct all the existence regions for each type of trajectory. It is then determined which trajectory (or which of several trajectories) will accomplish a

given turn Λ_Σ. If several trajectories exist at a given point, then it is necessary to select the shortest of them. Thereby, the answers will be given for all the questions of the formulated problem; the rotational angular velocities and their sequence, the number of rotation segments with the limiting allowable velocities, and their turn time will be determined.

Let us make the following remarks in conclusion. The condition (6.47) determines the following two relations, which permit one to find $\mathbf{p}_{E0}$ and $\mathbf{p}_{ET}$: the angle between the vectors $\boldsymbol{\lambda}_\Sigma$ and $\mathbf{p}_{E0}$ equals the angle between the vectors $\boldsymbol{\lambda}_\Sigma$ and $\mathbf{p}_{ET}$, and the dihedral angle between the planes, which are formed by the vectors $\boldsymbol{\lambda}_\Sigma$ and $\mathbf{p}_{E0}$ and $\boldsymbol{\lambda}_\Sigma$ and $\mathbf{p}_{ET}$, respectively, equals ϑ_Σ. On the other hand, we have seen that, on any optimum trajectory, the lengths of the component rotation arcs are uniquely determined. The position of the vectors $\mathbf{p}_{E0}$ and $\mathbf{p}_{ET}$ is then given uniquely. It would appear that the two conditions presented above are superfluous, or that the number of conditions for determining the position of the vectors $\mathbf{p}_{E0}$ and $\mathbf{p}_{ET}$ is greater than required. It is found that these two types of conditions determining the position of the vectors $\mathbf{p}_{E0}$ and $\mathbf{p}_{ET}$ are compatible on optimum trajectories. Hence, in particular, there follows the curious fact: the equality

$$\mathbf{p}_{E0} = \mathbf{p}_{ET},$$

occurs on the trajectory of the closed cycle VII, which, according to (6.47) is possible if and only if the vector $\boldsymbol{\lambda}_\Sigma$ is directed along the vector $\mathbf{p}_{E0}$.

One can see that a further increase of the number of trajectory segments leading to a double cycle is possible. In this case, if the first "rotation" of the vector $\mathbf{p}_E$ (i.e., the first cycle) is completed at time t_1, we have: / 274

$$p_E(t_1) = p_{E0},$$

and, consequently, the vector $\lambda_\Sigma(t_1)$ is also directed along the vector $\mathbf{p}_{E0}$. Since, for a double cycle, the resultant quaternion can be represented as:

$$\Lambda_\Sigma(T) = \Lambda_\Sigma(t_1) \circ \Lambda_\Sigma(T - t_1)$$

and $\Lambda_\Sigma(T)$ and $\Lambda_\Sigma(t_1)$ has the direction of $\mathbf{p}_{E0}$, then $\Lambda_\Sigma(T - t_1)$ also has the same direction and its rotation angle [as well as $\Lambda_\Sigma(t_1)$] equals half the angle ϑ_Σ.

We have an analogous partition for triple and quadruple cycles, etc. The limit of such cycles with their number increasing to infinity and each segment of motion decreasing to zero will be the sliding mode with all four velocities (in this case, $\omega_1, \omega_2, \omega_3, \omega_4$), which is equivalent to the stationary case of rotation with the velocity $\omega_{\Sigma 3}$.

Small rotations. The case of small rotations is of interest in that it permits one to find an analytic solution for any type of trajectory. We will carry out this investigation in detail in order to illustrate the example of the technique for constructing the regions for each type of trajectory.

We present the epxression for the quaternions, the solutions of the kinematic equations with the angular velocity values (6.44), in the order of their arrangement shown in Figure 6.1.

For the velocities ω_m, we have in accordance with (6.48) (we present only the first four solutions):

$$\left.\begin{aligned}
\Lambda_1 &= \cos\frac{\vartheta_1}{2} + \frac{e_1+e_2+e_3}{\sqrt{3}}\sin\frac{\vartheta_1}{2},\\
\Lambda_2 &= \cos\frac{\vartheta_2}{2} + \frac{-e_1+e_2+e_3}{\sqrt{3}}\sin\frac{\vartheta_2}{2},\\
\Lambda_3 &= \cos\frac{\vartheta_3}{2} + \frac{-e_1-e_2+e_3}{\sqrt{3}}\sin\frac{\vartheta_3}{2},\\
\Lambda_4 &= \cos\frac{\vartheta_4}{2} + \frac{e_1-e_2+e_3}{\sqrt{3}}\sin\frac{\vartheta_4}{2}.
\end{aligned}\right\} \qquad (6.54)$$

For the velocities $\omega_{\mathcal{A}n}$, we obtain the solutions in accordance with (6.50): / 275

$$\left.\begin{aligned}
\Lambda_{\mathcal{A}1} &= \cos\frac{\vartheta_{\mathcal{A}1}}{2} + e_1\sin\frac{\vartheta_{\mathcal{A}1}}{2},\\
\Lambda_{\mathcal{A}2} &= \cos\frac{\vartheta_{\mathcal{A}2}}{2} + e_2\sin\frac{\vartheta_{\mathcal{A}2}}{2},\\
\Lambda_{\mathcal{A}3} &= \cos\frac{\vartheta_{\mathcal{A}3}}{2} + e_3\sin\frac{\vartheta_{\mathcal{A}3}}{2}
\end{aligned}\right\} \qquad (6.55)$$

(we present the first three solutions).

For the velocities $\omega_{\mathcal{G}l}$, we present the solutions (6.49) for the first three values of the velocity:

$$\left.\begin{aligned}
\Lambda_{\mathcal{G}1} &= \cos\frac{\vartheta_{\mathcal{G}1}}{2} + \frac{e_2+e_3}{\sqrt{2}}\sin\frac{\vartheta_{\mathcal{G}1}}{2},\\
\Lambda_{\mathcal{G}2} &= \cos\frac{\vartheta_{\mathcal{G}2}}{2} + \frac{-e_1+e_3}{\sqrt{2}}\sin\frac{\vartheta_{\mathcal{G}2}}{2},\\
\Lambda_{\mathcal{G}3} &= \cos\frac{\vartheta_{\mathcal{G}3}}{2} + \frac{-e_2+e_3}{\sqrt{2}}\sin\frac{\vartheta_{\mathcal{G}3}}{2}.
\end{aligned}\right\} \qquad (6.56)$$

We will assume that the angles ϑ_k of all the component quaternions are small, so that their derivatives can be neglected. In this case, firstly, the angle of the resultant rotation ϑ_Σ will also be small; secondly, the multiplication of quaternions of small rotations will be performed by adding their vector parts (cf., Paragraph 3.1).

We will consider the trajectories of types IV — VII, which exist when the vector $\mathbf{p}_E$ is located in the region $\mathcal{A}_n$. Let us begin the investigation with the trajectory of type IV. In

accordance with (6.54), we obtain, for small angles ϑ_λ ;

$$\Lambda_\Sigma = \Lambda_1 \circ \Lambda_2 \approx 1 + \frac{e_1+e_2+e_3}{2\sqrt{3}}\vartheta_1 + \frac{-e_1+e_2+e_3}{2\sqrt{3}}\vartheta_2;$$

for the small angle ϑ_Σ, we also have

$$\Lambda_\Sigma \approx 1 + \frac{1}{2}(\gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3)\vartheta_\Sigma.$$

By equating the quaternion components, we have:

$$\gamma_1\vartheta_\Sigma = \frac{1}{\sqrt{3}}(\vartheta_1 - \vartheta_2), \quad \gamma_2\vartheta_\Sigma = \gamma_3\vartheta_\Sigma = \frac{1}{\sqrt{3}}(\vartheta_1 + \vartheta_2).$$

Hence, it is seen that the resultant quaternion can be positioned only on the arc of the sphere formed by its intersection with the plane for which $\gamma_2 = \gamma_3$. This should have been expected, since the sum of the two vectors θ_1 and θ_2 can give the resultant vector found in the plane formed by these vectors. By solving the obtained equalities, we find the magnitudes of the component rotation angles as a function of the quaternion Λ_Σ: / 276

$$\vartheta_1 = \frac{\sqrt{3}}{2}(\gamma_1 + \gamma_2)\vartheta_\Sigma, \quad \vartheta_2 = \frac{\sqrt{3}}{2}(\gamma_2 - \gamma_1)\vartheta_\Sigma.$$

The condition that these angles be positive defines the following segment of the arc:

$$\gamma_1 + \gamma_2 \geqslant 0, \quad \gamma_2 \geqslant \gamma_1, \quad \gamma_2 = \gamma_3. \tag{6.57}$$

It is on this arc that the end of the vector of the quaternion Λ_Σ can be positioned for this type of trajectory.

Let us consider the trajectory of type V. By analogy to the preceding case, we have the expression for the resultant quaternion:

$$\Lambda_\Sigma = \Lambda_1 \circ \Lambda_2 \circ \Lambda_3 \approx 1 + \frac{e_1+e_2+e_3}{2\sqrt{3}}\vartheta_1 + \\ + \frac{-e_1+e_2+e_3}{2\sqrt{3}}\vartheta_2 + \frac{-e_1-e_2+e_3}{2\sqrt{3}}\vartheta_3.$$

from which:

$$\gamma_1\vartheta_\Sigma=\frac{1}{\sqrt{3}}(\vartheta_1-\vartheta_2-\vartheta_3),$$
$$\gamma_2\vartheta_\Sigma=\frac{1}{\sqrt{3}}(\vartheta_1+\vartheta_2-\vartheta_3),$$
$$\gamma_3\vartheta_\Sigma=\frac{1}{\sqrt{3}}(\vartheta_1+\vartheta_2+\vartheta_3).$$

We solve the equation for the angles ϑ_k:

$$\vartheta_1=\frac{\sqrt{3}}{2}\vartheta_\Sigma(\gamma_1+\gamma_2),$$
$$\vartheta_2=\frac{\sqrt{3}}{2}\vartheta_\Sigma(\gamma_2-\gamma_1),$$
$$\vartheta_3=\frac{\sqrt{3}}{2}\vartheta_\Sigma(\gamma_3-\gamma_2).$$

The condition that the rotation angles ϑ_k be positive and / 277
the condition (6.51) define the following region on the sphere, in which the considered motion consisting of three segments exists:

$$\left.\begin{array}{ll}\gamma_1+\gamma_3\geqslant 0, & \gamma_2\geqslant\gamma_1,\quad \gamma_3\geqslant\gamma_2,\\ \gamma_2-\gamma_1\geqslant\gamma_1+\gamma_3, & \gamma_2-\gamma_1\geqslant\gamma_3-\gamma_2.\end{array}\right\} \qquad (6.58)$$

Obviously, each of these inequalities defines a boundary, which is an intersection of the sphere with the planes

$$\gamma_1+\gamma_3=0,\ \gamma_2=\gamma_1,\ \gamma_3=\gamma_2,\ \gamma_2=2\gamma_1+\gamma_3,\ 2\gamma_2=\gamma_1+\gamma_3,$$

i.e., all the boundaries are arcs of a great circle (all the planes pass through the center of the sphere). These five boundaries are shown in Figure 6.5, where the hemisphere has been expanded onto the plane. The inequalities (6.58) isolate the shaded region. As is seen from the figure, there are only four boundaries [these are the first, third, fourth, and fifth conditions of (6.58)]. The first three conditions of (6.58) are identical to the conditions of (6.57) and we define the boundary, with the transition to which one of the three motion segments

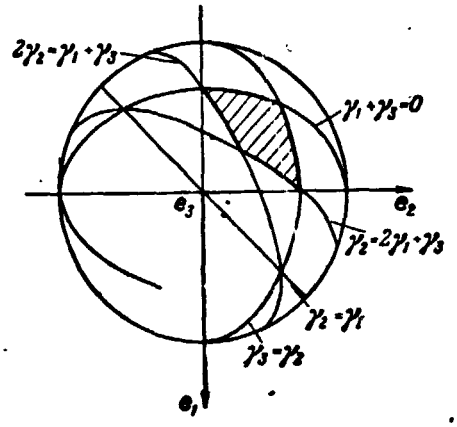

Figure 6.5.

vanishes. Hence, it follows that the boundaries of the region of the trajectory of type V on one side are lines of the type (6.57), i.e., lines of the trajectory of the type IV. It should be expected that the other two boundaries [the fourth and fifth conditions of (6.58)] will define a transition to the trajectories of type VI. The strict inequalities of (6.51) are satisfied everywhere inside the region of the trajectories of type V, and the angles ϑ_1, ϑ_2 and ϑ_3 are positive. / 278

Let us turn to the trajectories of type VI. By substituting the magnitudes of the quaternions (6.54) into the expression for the resultant quaternion, we obtain:

$$\Lambda_\Sigma=\Lambda_1\circ\Lambda_2\circ\Lambda_3\circ\Lambda_4\approx 1+\frac{e_1+e_2+e_3}{2\sqrt{3}}\vartheta_1+\frac{-e_1+e_2+e_3}{2\sqrt{3}}\vartheta_2+$$
$$+\frac{-e_1-e_2+e_3}{2\sqrt{3}}\vartheta_3+\frac{e_1-e_2+e_3}{2\sqrt{3}}\vartheta_4.$$

By equating the elements for the units $\mathbf{e}_i$, we obtain:

$$\gamma_1\vartheta_\Sigma=\frac{1}{\sqrt{3}}(\vartheta_1-\vartheta_2-\vartheta_3+\vartheta_4),$$
$$\gamma_2\vartheta_\Sigma=\frac{1}{\sqrt{3}}(\vartheta_1+\vartheta_2-\vartheta_3-\vartheta_4),$$
$$\gamma_3\vartheta_\Sigma=\frac{1}{\sqrt{3}}(\vartheta_1+\vartheta_2+\vartheta_3+\vartheta_4),$$

and the second condition of (6.52) gives another equality:

$$\vartheta_2=\vartheta_3,$$

thanks to which, these equations can be solved for the component rotation angles:

$$\vartheta_1=\frac{\sqrt{3}}{4}\vartheta_\Sigma(2\gamma_2+\gamma_1+\gamma_3),$$
$$\vartheta_2=\vartheta_3=\frac{\sqrt{3}}{4}\vartheta_\Sigma(\gamma_3-\gamma_1),$$
$$\vartheta_4=\frac{\sqrt{3}}{4}\vartheta_\Sigma(\gamma_1+\gamma_3-2\gamma_2).$$

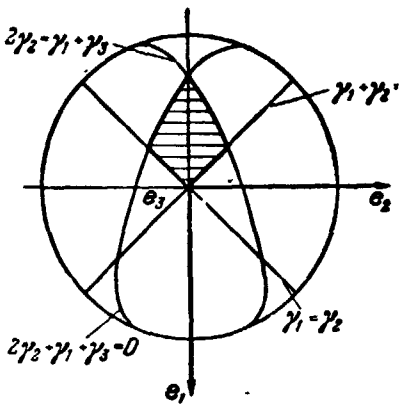

Figure 6.6.

The requirement that the angles ϑ_1, ϑ_2, ϑ_3 и ϑ_4, be positive and the conditions (6.25) define the following boundaries of the region for the trajectories of type VI:

$$\left.\begin{array}{l} 2\gamma_2 \geqslant \gamma_1+\gamma_3,\quad 2\gamma_2+\gamma_1+\gamma_3 \geqslant 0,\quad \gamma_3 \geqslant \gamma_1, \\ \gamma_1+\gamma_2 \leqslant 0,\quad \gamma_1 \geqslant \gamma_2. \end{array}\right\} \quad (6.59)$$

This region and the boundaries (6.59) are shown in Figure 6.6. It is seen that the first group of conditions (6.59) define the boundaries [cf., (6.58)] with the trajectories of the type V, while the second group specifies the boundary with trajectories of the same type, which consist of another sequence of rotations. / 279

The last statement follows from the fact that trajectories of type VII are equivalent to trajectories of type VI, since small rotations can be permuted:

$$\Lambda_\Sigma=\Lambda_1\circ\Lambda_2\circ\Lambda_3\circ\Lambda_4\circ\Lambda_1'=\Lambda_1\cdot\Lambda_1'\circ\Lambda_2\circ\Lambda_3\circ\Lambda_4,$$

and this trajectory coincides with trajectories of the type VI, with consideration of the fact that the first rotation angle equals $\vartheta_1+\vartheta_1'$.

The final picture for the regions of the trajectories of the type IV — VII is presented in Figure 6.7. The following notations are used in the figure: each region and boundary is denoted by the corresponding trajectory; however, the index of some quaternions is denoted by a bold-face digit. This denotes that the rotation angle of the given quaternion is less than the limiting possibility; for example, $\Lambda_2\circ\Lambda_3\circ\Lambda_4$ indicates that $\vartheta_2<\vartheta_3$ and $\vartheta_4<\vartheta_3$. The boundaries of the regions correspond to an additional condition, which decreases the number of free

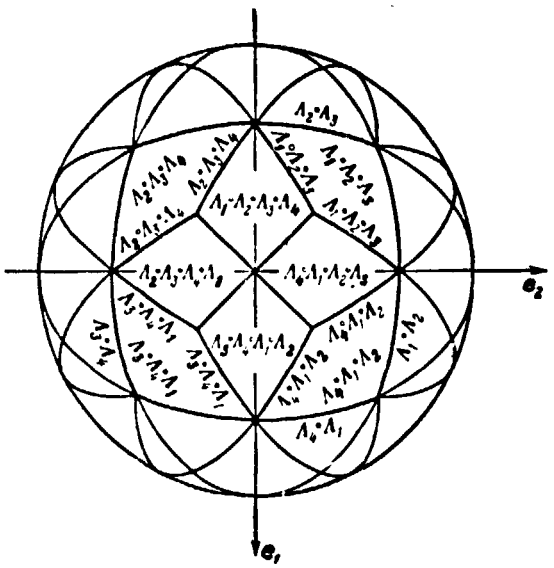

Figure 6.7.

parameters to two, which is reflected in the notation; thus, / 280 the boundary $\Lambda_2 \circ \Lambda_3 \circ \Lambda_4$ corresponds to the trajectory for which $\vartheta_2 = \vartheta_3 > \vartheta_4$; it separates the regions $\Lambda_2 \circ \Lambda_3 \circ \Lambda_4 \circ \Lambda_1$ and $\Lambda_2 \circ \Lambda_3 \circ \Lambda_4$. Figure 6.7 permits illustration of all the statements relating to the regions and boundaries of the motion, which were expressed in the qualitative investigation.

The regions of motion presented in Figure 6.7 correspond to trajectories obtained for motion of the vector $\mathbf{p}_E$ in the region $\mathcal{A}_3$, when it "avoids" the point $\mathbf{e}_3$ (cf., Figure 6.2). Exactly the same trajectories and regions for the trajectories of type IV — VIII corresponding to the picture exist in any region $\mathcal{A}_n$. In any case, it is sufficient to construct the partition into regions of motion just in the zone, for example, $\mathcal{A}_3\mathcal{C}_1\mathcal{B}_2\mathcal{C}_2$ (Figure 6 8) or even in half of it, $\mathcal{A}_3\mathcal{C}_1\mathcal{B}_2$, since all the remaining zones of the sphere are adequately arranged with respect to the trihedron and the rotational angular velocities and, consequently, have the same partition into regions of motion as in the zone $\mathcal{A}_3\mathcal{C}_1\mathcal{B}_2\mathcal{C}_3$.

It can be seen that the regions for the trajectories of the / 281 type V and VI and the boundaries IV shown in Figure 6.7 completely cover just this zone. Thus, motion of the types IV, V, and VI have regions completely covering the whole sphere. Hence, it follows that the trajectories of type IV, V, VI permit an arbitrary small rotation to be performed.

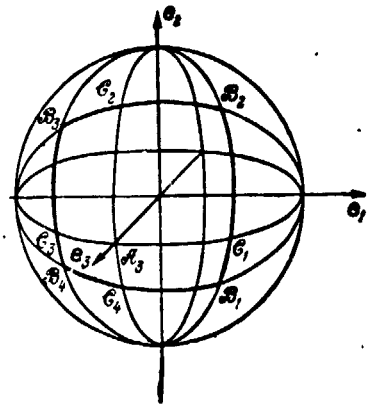

Figure 6.8.

Let us find the time for performing the turn. The magnitude of the angular velocity ω_m equals $\sqrt{3}$. Hence, it follows that the time of the motion will equal the sum of all the angles of the component motions divided by $\sqrt{3}$. By using the obtained values of the angles for each type of trajectories IV — VII, we obtain:

$$\left.\begin{aligned} T_{IV} &= \frac{\vartheta_1+\vartheta_2}{\sqrt{3}} = \vartheta_\Sigma\gamma_2 = \\ &= \vartheta_\Sigma\gamma_3, \\ T_V &= \frac{\vartheta_1+\vartheta_3+\vartheta_5}{\sqrt{3}} = \\ &= \vartheta_\Sigma\gamma_3, \\ T_{VI} &= T_{VII} = \\ &= \frac{\vartheta_1+\vartheta_2+\vartheta_3+\vartheta_4}{\sqrt{3}} = \vartheta_\Sigma\gamma_3, \end{aligned}\right\} \tag{6.60}$$

i.e., for all trajectories, the time of motion for the same angle ϑ_Σ is maximum and equals ϑ_Σ when Λ_Σ is located at the points $\mathscr{A}_n$, and uniformly decreases to the value $\vartheta_\Sigma/\sqrt{2}$ at the points $\mathscr{C}_l$ and to the value $\vartheta_\Sigma/\sqrt{3}$ at the points $\mathscr{B}_m$.

Let us now consider the other possible trajectories. The stationary trajectories of the types I — III correspond to points on the sphere (6.48) — (6.50). It can be seen that the time of motion along these trajectories coincides with the time of motion along the trajectories of the type IV — VII. In this sense, all these trajectories are optimum and the motion is possible along both types of trajectories.

The trajectory of the type VIII and the trajectories of the types IX and X equivalent to it (in view of the cummutativity of small rotations) determine for the small angles θ_1 and $\theta_{\gamma 1}$ the motion [in accordance with (6.54) and (6.56)]: / 282

$$\Lambda_\Sigma = \Lambda_1 \circ \Lambda_{\gamma 1} \approx 1 + \frac{e_1+e_2+e_3}{2\sqrt{3}}\theta_1 + \frac{e_1+e_2}{2\sqrt{2}}\theta_{\gamma 1},$$

$$\gamma_1\theta_\Sigma = \gamma_2\theta_\Sigma = \frac{1}{\sqrt{3}}\theta_1 + \frac{1}{\sqrt{2}}\theta_{\gamma 1},\ \gamma_3\theta_\Sigma = -\frac{1}{\sqrt{3}}\theta_1,$$

from which we have:

$$\theta_1 = \sqrt{3}\theta_\Sigma\gamma_3 \geqslant 0,\quad \theta_{\gamma 1} = \sqrt{2}\theta_\Sigma(\gamma_2 - \gamma_3) = \sqrt{2}\theta_\Sigma(\gamma_1 - \gamma_3) \geqslant 0.$$

Such trajectories exist on the following curve on the sphere:

$$\gamma_2 = \gamma_1 \geqslant \gamma_3$$

and the time of motion along these trajectories will equal:

$$T_{VIII} = T_{IX} = T_X = \frac{\theta_1}{\sqrt{3}} + \frac{\theta_{\gamma 1}}{\sqrt{2}} = \theta_\Sigma\gamma_1 = \theta_\Sigma\gamma_2.$$

Since the region of existence of such trajectories is just the boundary of the symmetric zones (curve $\mathcal{B}_2\mathcal{B}_1$ in Figure 6.8), it can be seen that the trajectory of the type IV is possible on this boundary with exactly the same time of motion.

We will consider the trajectory of type XI. For small angles, in accordance with (6.54) and (6.56), we have:

$$\Lambda_\Sigma = \Lambda_1 \circ \Lambda_{\gamma 1} \circ \Lambda_2 \approx 1 + \frac{e_1+e_2+e_3}{2\sqrt{3}}\theta_1 + \frac{-e_1+e_2+e_3}{2\sqrt{3}}\theta_2 + \frac{e_1+e_2}{2\sqrt{2}}\theta_{\gamma 1},$$

from which

$$\gamma_1\theta_\Sigma = \frac{1}{\sqrt{3}}(\theta_1 - \theta_2) + \frac{1}{\sqrt{2}}\theta_{\gamma 1},$$

$$\gamma_2\theta_\Sigma = \frac{1}{\sqrt{3}}(\theta_1 + \theta_2) + \frac{1}{\sqrt{2}}\theta_{\gamma 1},$$

$$\gamma_3\theta_\Sigma = \frac{1}{\sqrt{3}}(\theta_1 + \theta_2).$$

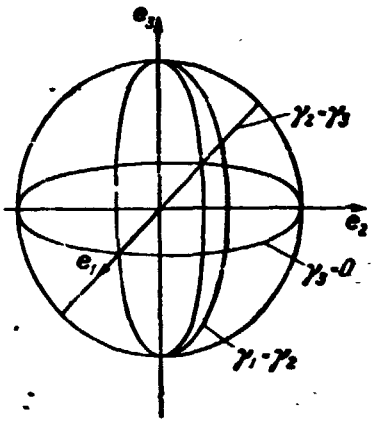

Figure 6.9.

We solve these equalities for the rotation angles / 283

$$\theta_1=\sqrt{3}\theta_\Sigma(\gamma_3+\gamma_1-\gamma_2)=\sqrt{3}\theta_\Sigma\gamma_3,$$
$$\theta_2=\sqrt{3}\theta_\Sigma(\gamma_2-\gamma_1)=0,$$
$$\theta_{\Sigma 1}=\sqrt{2}\theta_\Sigma(\gamma_2-\gamma_3).$$

The region of existence of this trajectory is determined by the conditions

$$\gamma_3\geqslant 0,\quad \gamma_2=\gamma_1,$$
$$\gamma_2\geqslant\gamma_3.$$

This region is shown in Figure 6.9. As is seen, it is the boundary of the region for the trajectory of type V (this is the region $\Lambda_8\circ\Lambda_1\circ\Lambda_2$, obtained for motion of the vector $\mathbf{p}_E$ in the region $\mathcal{A}_2$). The time of motion along the trajectory of type IX equals:

$$T_{XI}=$$
$$=\frac{\theta_1+\theta_2}{\sqrt{3}}+\frac{\theta_{\Sigma 1}}{\sqrt{2}}=\theta_\Sigma\gamma_3,$$

i.e., it coincides with the time of motion along the trajectory of type V [cf., (6.60)].

The trajectories of types XII — XIV do not exist for small angles, since $\theta_2=\frac{2\pi}{3}$ for them.

Thus, all the regions of existence for the optimum turns for small angles have been found.

Let us now answer the following question: how many of these optimum equations differ from the control by quaternion components (6.37)? Let Λ_Σ be located in the region bounded by the heavy arcs (cf., Figure 6.7). It can be seen that this case corresponds to the passage of the vector λ_Σ through one face of the cube (6.43). In this case, the magnitude of the

maximum angular velocity will equal $\omega_{x3}/\gamma_3 = 1/\gamma_3$ for the control (6.37). Hence, it follows that the time for a turn by the angle Φ_x with such control will equal $T = \gamma_3 \Phi_x$, i.e., it coincides with the time of the optimum turn (6.60). / 284

Thus, in the first approximation for small turn angles, the control (6.37) is optimum even for limitations of the type (6.44). If quantities of second order are taken into account, then the considered optimum turns are "shorter," nevertheless, than rotation of the type (6.37). It will be shown below that this difference becomes significant for large angles.

Arbitrary turn angles. As was shown in the example of small rotation angles, the process for solving the problem reduces to constructing regions of existence on the sphere for each type of trajectory and determining the sufficient conditions for optimization. The case when the component rotations are arbitrary angles is significantly more complicated and cannot be solved analytically. In any case, one can set up the equations expressing the quantities $\lambda_{\Sigma i}$ (i = 0, 1, 2, 3) of the resultant quaternion as a function of the component rotation angles for each type of trajectory. However, inversion of these equalities cannot be performed in the arbitrary case. To solve the problem, we have used the method of constructing the region boundaries. Each region boundary is determined so that the first and last rotation have their maximum or minimum value; the number of free parameters does not exceed two for any type of motion. The solution was constructed by computer calculation of the following problem: one parameter (the angle of one rotation) takes a fixed value; the second is chosen so as to ensure the required angle of the resultant turn Φ_x; this defines one point on the sphere. By varying the first parameter (the angle of the first rotation) from the minimum to the maximum value, we obtain a line on the

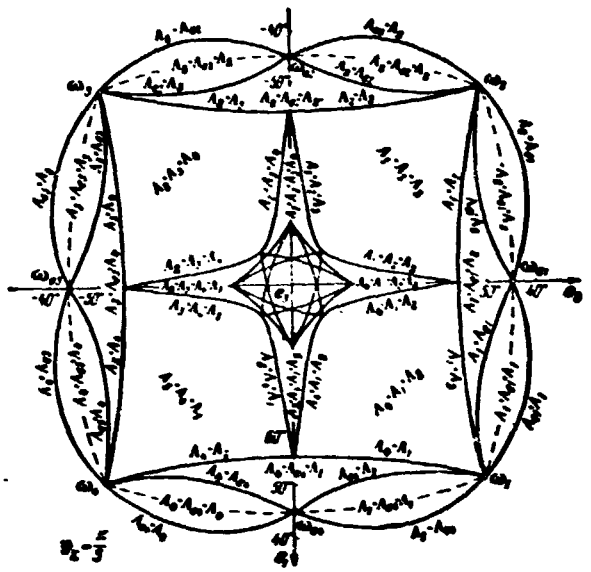

Figure 6.10.

sphere, the boundary of the region. The calculations were performed for four values of the angle θ_2, equaling $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$ and π. The results of the solution are presented in Figures 6.10 — 6.15 in a zone on the sphere with notations analogous to those shown in Figure 6.7. In all the figures, the hemisphere is expanded onto the plane; the dashed line represents the boundaries of symmetric zones.

The zones for the case when the angle $\theta_2=\frac{\pi}{3}$ are shown in Figure 6.10. As is seen from the figure, the largest regions for such rotation exist for the position of the vector $\mathbf{p}_{E0}$ in the zone $\mathcal{A}_3$ and, in particular, for trajectories of the type V. The boundaries of these regions, which are shown in the figure, consist of the trajectories of type IV on one side and separate this region from the trajectories of type VI on the other side. The trajectories of type VI occupy narrow regions along the coordinate axes; their boundaries with the trajectories of the type VII are located closer to the center ($\mathbf{e}_3$); the closed cycle

/ 285

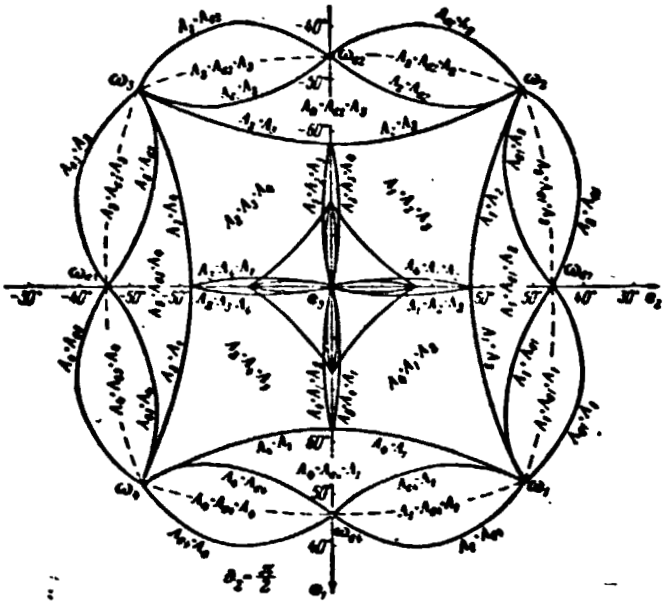

Figure 6.11.

trajectory is also shown. As is seen from the figure, inside and on the boundary of the cycle there exist simultaneously the trajectory of type VI and the trajectory consisting of a large number of segments (of the type VII, closed cycle, etc.). However, the calculations have shown that the following hypothesis appears valid on the whole:

4. Trajectories consisting of a smaller number of segments are optimum.

Some region outside the trajectories of type V is occupied / 287 by trajectories obtained when the vector $\mathbf{p}_{E0}$ is located on the separatrix, i.e., trajectories of the type X and XI, which are bounded by the trajectories of the type VIII, IX, and IV. It can be seen that a continuous transition from one type of motion to another is accomplished everywhere.

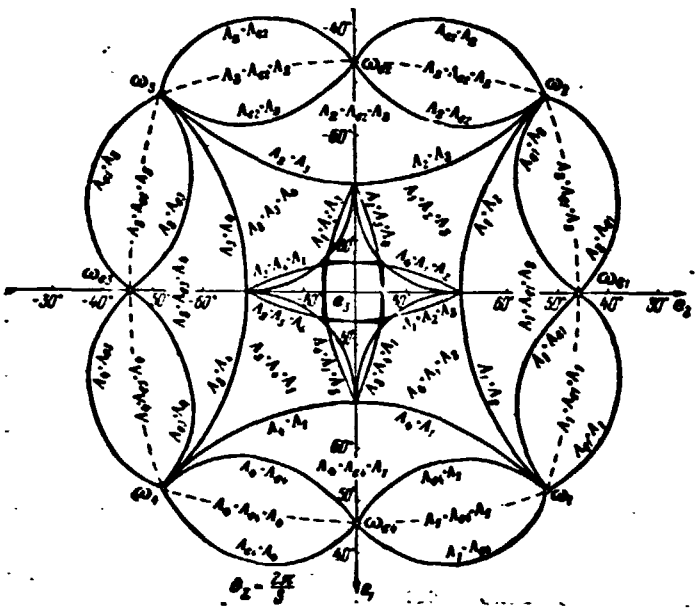

Figure 6.12.

A decrease of the regions determined by the motion of the vector $\mathbf{p}_E$ in the zone $\mathcal{A}_3$ and a broadening of the regions of the separatrix (we note that the latter region was completely absent for small angles) occur with increasing rotation angle ϑ_Σ (cf., Figure 6.11 for $\vartheta_\Sigma=\frac{\pi}{2}$ and Figure 6.12 for $\vartheta_\Sigma=\frac{2\pi}{3}$). At the same time, the regions for the trajectories of the type VI begin to decrease, and then expand again. The angle $\vartheta_\Sigma=\frac{2\pi}{3}$ is a limiting angle, for which the region of the trajectories of type V approaches the points ω_m. On the other hand, trajectories of the types XII — XIV do not exist for $\vartheta_\Sigma\leqslant\frac{2\pi}{3}$. / 288

The trajectories for the angle $\vartheta_\Sigma=\pi$ occupy a peculiar place. The regions obtained when the vector $\mathbf{p}_{E0}$ is located in the zone $\mathcal{A}_3$ are presented in Figure 6.13. As is seen in the figure, a further expansion of the zones of the trajectories of type VI and a reduction of the regions of the trajectories of type V have taken place; the zones of the regions of the trajectories of type VII have also appeared. The closed cycle trajectory has

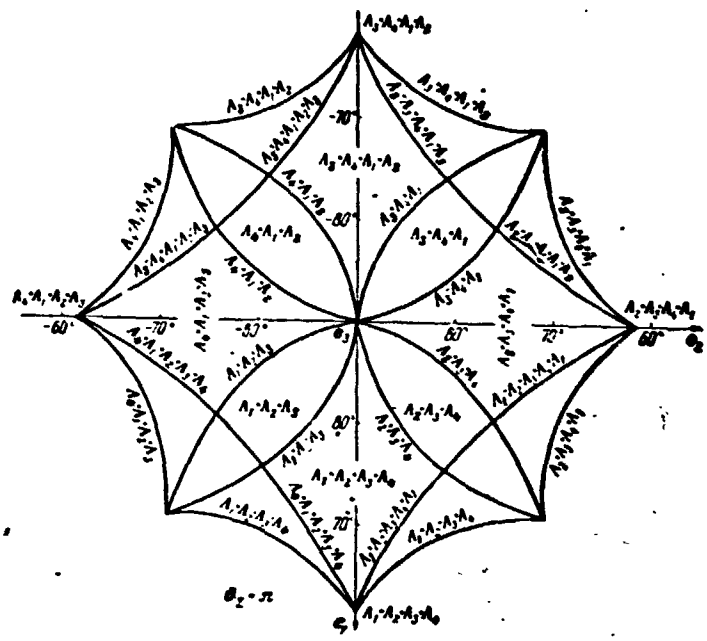

Figure 6.13.

become maximum; however, the region of this type of trajectory as a whole has been quite reduced. On the other hand, the regions of trajectories obtained with the motion of the vector $\mathbf{p}_E$ along the separatrix are shown in Figure 6.14; as is seen from the figure, these regions cover the whole sphere. The regions of the trajectories of types XIII and XIV are not shown in this figure (one of the boundaries of these trajectories is shown in Figure 6.15) since it has been shown that, firstly, the zone of their existence is completely covered by the region of the trajectories of the type XI, and, secondly, the trajectories of type XI are "shorter" than the trajectories of types XIII and XIV, i.e., hypothesis 4 is valid for this type of trajectory also. / 289

The regions of all the types of trajectories are shown in Figure 6.15. It is seen from this figure that, at the center of the region $\mathcal{A}_3$, there exist simultaneously the regions of the

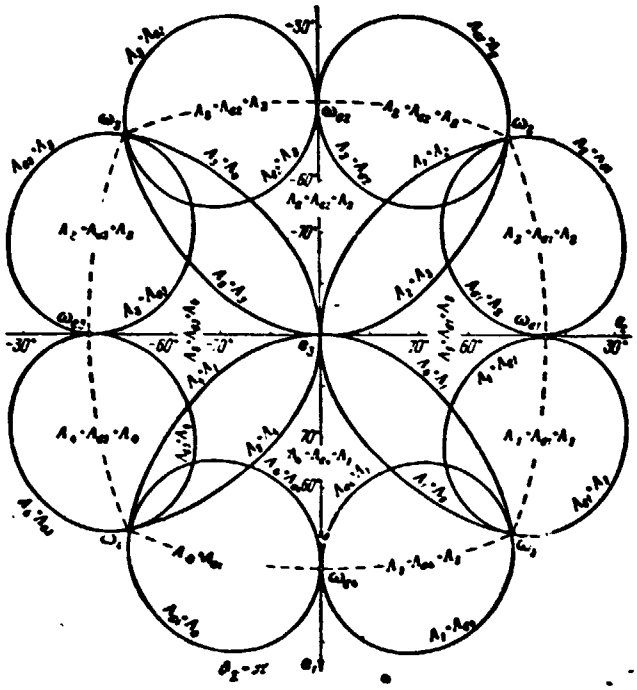

Figure 6.14.

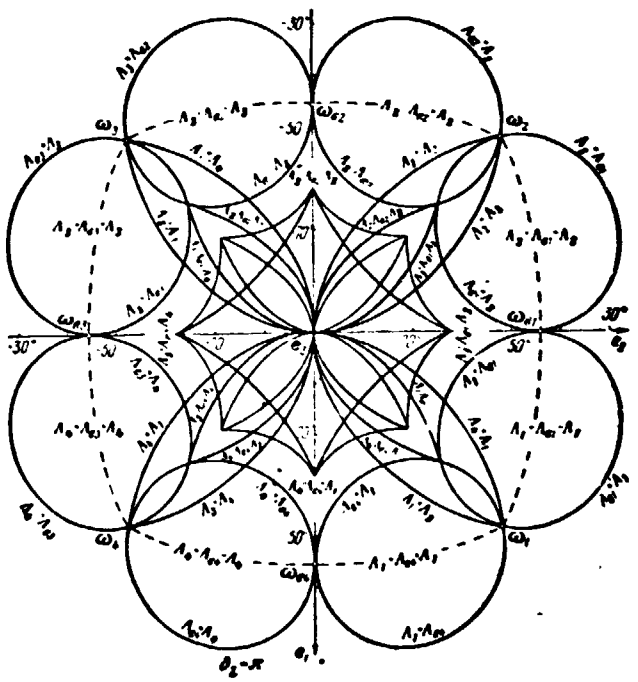

Figure 6.15.

trajectories of the types V — VII and VIII — XI. The calculations have shown that the trajectories of types VIII — XI are "shorter" than the remaining trajectories everywhere. Thus, / 290 one can obtain the optimum motion, which executes any given turn.

Let us consider in conclusion the question about how much the constructed optimum control differs from the control of the type (6.37) by the quaternion components. One can see that the optimum control exactly coincides with (6.37) at the points ω_m and $\omega_{\gamma l}$ in all cases; analysis has shown that, at the remaining points, the optimum control is "shorter." The difference between the times of the turn with the control (6.37) and the optimum turn reaches a maximum (for any fixed angle ϑ_Σ) at the points $\mathcal{A}_n$. This difference at these points, as a function of the angle ϑ_Σ, increases with an increase of this angle, reaching a maximum for $\vartheta_\Sigma = \pi$. The optimum turn, in this case, is determined by the trajectories of type IV, for which both component rotations have the angle $\frac{2\pi}{3}$, i.e., the total rotation angle equals 240°. However, since the rotation occurs with the velocity $|\omega_m| = \sqrt{3}$ then the turn time:

$$T_\Sigma = \frac{240^\circ}{\sqrt{3}} \approx 138^\circ,8 < 180^\circ.$$

§ 6.5. DYNAMIC OPTIMUM TURN. FORMULATION OF THE PROBLEM

In the problem of dynamic optimum turn, it is assumed that the magnitude of the external torque applied to the body is the control function. The equations of motion of the rigid body are the dynamic equations supplemented by the kinematic equations. We will consider rotational motion of a rigid body relative to a fixed coordinate system I, whose origin is located at the center of mass of the body. The Euler equations (5.51) will be the dynamic equations of motion.

We will assume that the control $\mathbf{M}\{M_1, M_2, M_3\}$ is limited and belongs to some closed region $\mathbf{M}$:

$$\mathbf{M}\{M_1, M_2, M_3\} \in \mathbf{M}. \tag{6.61}$$

The magnitudes of the angular velocity determined by Equation (5.51) specify the kinematics of the rigid body motion described by Equations (6.1). For a complete description of the motion, it is necessary to specify the boundary conditions of the motion / 291 on the position of the body:

$$\Lambda(0) = \Lambda_0, \quad \Lambda(T) = \Lambda_T, \tag{6.62}$$

where 0 and T are the times of the beginning and end of the motion. The boundary conditions on the velocity are the following:

$$\omega(0) = \omega_0, \quad \omega(T) = \omega_T, \tag{6.63}$$

where ω_0 and ω_T are some desired allowable [in the sense that they can be achieved under the limitation of Equation (6.61)] values.

The problem, in which

$$\omega_0 = \omega_T = 0, \tag{6.64}$$

has practical value, i.e., the problem of a body turning from one fixed position to another.

In order to completely close the problem, it is necessary to introduce the criterion for optimization. We will consider two formulations of the problem: the dynamic optimum turn with respect to the speed of response and the optimum turn with respect to the so-called "consumption" value. In the first case, the time T of the turn is minimized; in the second case for a

fixed turn time T, the following quantity is minimized:

$$G=\int_0^T\left(\frac{|M_1|}{r_1}+\frac{|M_2|}{r_2}+\frac{|M_3|}{r_3}\right)dt, \qquad (6.65)$$

where r_1, r_2, and r_3 are certain lengths, which have the meaning of the moment arm of the rocket thrust producing the corresponding torque M_i. The quantity G equals the total impulse of the rocket thrust and determines the consumption of fuel expended by the control system in performing the turn. In the particular case $r_i = 1$, the quantity G determines the total angular impulse, obviously, a formulation of the problem when this quantity is minimized is possible.

The dynamic equations (5.51), together with (6.1) under the given initial conditions (6.62), (6.63), and for some control functions $M_i(t)$, have a unique solution (trajectory). Obviously, the trajectory does not arrive at the final point determined by / 292 Λ_T and ω_T [cf., (6.62) and (6.63)] for all possible control functions. We will assume that there always exists a subset of control functions $M_i(t)$ belonging to the set (6.61), for which the trajectory of the system motion satisfies the boundary conditions (6.62) and (6.63). The problem consists of finding, from all the possible allowable controls, the one which has the minimum value of the selected control quality criterion: the time T or impulse G. To find the conditions for optimizing the control, we use the formalism of Pontryagin's principle of the maximum [6, 41].

In order to have the same approach to the solution in both formulations of the optimization problem, we will reduce the case of optimization with respect to impulse to the case of optimization with respect to the speed of response by adding to the equations of motion the supplementary equation:

$$\dot{G} = \frac{|M_1|}{r_1} + \frac{|M_2|}{r_2} + \frac{|M_3|}{r_3}. \qquad (6.66)$$

obtained from the optimization criterion (6.65). Obviously, the initial value of G equals zero; the final value is not fixed and the equation for it must be minimized.

We will introduce the supplementary functions φ_1, φ_2 and φ_3, corresponding to the variables ω_1, ω_2 and ω_3, the functions ψ_i ($i = 0, 1, 2, 3$) corresponding to the coordinates λ_i, and the function g corresponding to G. With the help of these functions, we will construct the Hamiltonian:

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_д + \mathcal{H}_к + \mathcal{H}_р = \langle \varphi \dot{\omega} \rangle + \langle \dot{\lambda} \psi \rangle + \langle \dot{G} g \rangle = \\ &= \varphi_1 \left(\frac{M_1}{J_1} + \frac{J_2 - J_3}{J_1} \omega_2 \omega_3 \right) + \varphi_2 \left(\frac{M_2}{J_2} + \frac{J_3 - J_1}{J_2} \omega_1 \omega_3 \right) + \\ &+ \varphi_3 \left(\frac{M_3}{J_3} + \frac{J_1 - J_2}{J_3} \omega_1 \omega_2 \right) + \mathcal{H}_к + g \left(\frac{|M_1|}{r_1} + \frac{|M_2|}{r_2} + \frac{|M_3|}{r_3} \right).\end{aligned} \qquad (6.67)$$

where the Hamiltonian is divided into the "dynamic" $\mathcal{H}_д$, the "kinematic" $\mathcal{H}_к$, and "consumption" $\mathcal{H}_р$.

The quantity $\mathcal{H}_к = \frac{1}{2}(\omega_1 p_1 + \omega_2 p_2 + \omega_3 p_3)$ is the Hamiltonian considered in paragraph 6.2 for the kinematic optimum turn problem (6.6); it can be expressed in terms of the kinematic conjugate control vector. In the case of control with respect to the speed of response, the last term $\mathcal{H}_р$ is missing in the relation (6.67). / 293

The supplementary variables are defined by the equations:

$$\dot{\varphi}_k = -\frac{\partial \mathcal{H}}{\partial \omega_k} (k = 1, 2, 3), \quad \dot{\psi}_i = -\frac{\partial \mathcal{H}}{\partial \lambda_i} (i = 0, 1, 2, 3), \quad \dot{g} = -\frac{\partial \mathcal{H}}{\partial G}.$$

We write these equations separately for the different groups of variables. We have the following equations defining the functions φ_k:

$$\left.\begin{aligned}
\dot{\varphi}_1 &= \varphi_2 \frac{J_1 - J_3}{J_2}\omega_3 + \varphi_3 \frac{J_2 - J_1}{J_3}\omega_2 - \frac{1}{2}p_1,\\
\dot{\varphi}_2 &= \varphi_1 \frac{J_3 - J_2}{J_1}\omega_3 + \varphi_3 \frac{J_2 - J_1}{J_3}\omega_1 - \frac{1}{2}p_2,\\
\dot{\varphi}_3 &= \varphi_1 \frac{J_3 - J_2}{J_1}\omega_2 + \varphi_2 \frac{J_1 - J_3}{J_2}\omega_1 - \frac{1}{2}p_3,
\end{aligned}\right\} \qquad (6.68)$$

where the quantities p_k are the quantities introduced previously (6.15), the projections of the conjugate vector $\mathbf{p}_{\Phi}$.

Since the kinematic variables λ_i (i = , 2, 3) appear only in the kinematic Hamiltonian $\mathcal{H}_к$

$$\dot{\psi}_i = -\frac{\partial \mathcal{H}}{\partial \lambda_i} = -\frac{\partial \mathcal{H}_к}{\partial \lambda_i},$$

then the equations for the variable ψ_i will be the Equations (6.8) and (6.13) obtained previously for the kinematic problem.

Finally, for the case of optimization with respect to the rocket thrust impulse, we have the last equation for the variable g:

$$\dot{g} = 0. \qquad (6.69)$$

This equation is missing in the case of optimization with respect to the turn time.

The necessary conditions for optimizing the control involve the fact that the control M_k (k = 1, 2, 3) must maximize the Hamiltonian $\mathcal{H}$ on the set (6.61) of allowable control. Since the control appears only in the first ("dynamic") and third ("consumption") terms of the Hamiltonian, then this condition is written in the form: / 294

$$\begin{aligned}\mathcal{H}\{M_k(t),\ \omega_k(t),\ \lambda_i(t),\ G(t),\ \varphi_k(t),\ \psi_i(t),\ g(t)\} &= \\ = \max_{M_k \in M} \mathcal{H}\{M_k,\ \omega_k(t),\ \lambda_i(t),\ G(t),\ \varphi_k(t),\ \psi_i(t),\ g(t)\} &= \\ = \max_{M_k \in M} \mathcal{H}_д(\omega_k,\ \varphi_k,\ M_k) + \max_{M_k \in M} \mathcal{H}_р(M_k,\ g,\ G).\end{aligned} \tag{6.70}$$

With the formulation of the problem for minimum consumption, when the turn time T is fixed, it is necessary to supplement these conditions by the following relation, which must be satisfied at the final moment of time T (cf., [6], paragraph 8, theorem 6):

$$g(T) = -1. \tag{6.71}$$

The written relations are sufficient for solving (in principle) the formulated problem. Actually, the condition for the maximum (6.70) determines the control as a function of ω_k and the supplementary variables φ_k and g. By formally expressing the control function M_k, with the help of the conditions for the maximum, by φ_k and g, we obtain a closed system of fourteen differential equations of first order (6.51), (6.68), (6.1), (6.8) (not taking into account the equations for G and g). These differential equations in turn determine the functions ω_i, λ_i, φ_k, ψ_i, g $(i = 0, 1, 2, 3;\ k = 1, 2, 3)$ with the necessary number of boundary conditions (6.62), (6.63), and (6.71). The constants specifying the functions φ_k and ψ_i should be selected such that the trajectory $\{\omega_k, \lambda_i\}$ emerging from the initial point (6.62), (6.63) passes through the specified final point.

In the case of optimizing control with respect to the speed of response, the control M_k (k = 1, 2, 3) is determined by the condition for the maximum of only the dynamic component of the Hamiltonian:

$$\max \mathcal{H} = \max_{M_k \in M} \mathcal{H}_д(M_k,\ \omega_k,\ \varphi_k). \tag{6.72}$$

This itself specifies the control as a function of the quantities ω_k and φ_k, and the form of this dependence is determined by the configuration of the set of controls $\mathbf{M}$.

In the case of optimizing the control with respect to the thrust impulse, we have, in accordance with (6.69) and (6.71): / 295

$$g(t) = -1.$$

Hence, the selection of the optimum control will be found from the control for the maximum of the function:

$$\max \mathcal{H} = \max_{M_k \in \mathbf{M}} \{\mathcal{H}_k (M_k, \omega_k, \varphi_k) - \frac{|M_1|}{r_1} - \frac{|M_2|}{r_2} - \frac{|M_3|}{r_3} \qquad (6.73)$$

and will be determined by more complicated relations depending on the turn time T (this will be shown below).

The systems of Equations (5.51) and (6.68) are systems of nonlinear equations; thus, even when a direct representation of the control M_k is obtained as a function of φ_k and ω_k, the solution of the equation cannot be obtained. Moreover, if one proceeds from the condition (6.73), then its solution is possible only for known (found) functions φ_k and ω_k. One should also take into account that the quantities p_k determined as a function of the solution of the kinematic equation, which also cannot be represented in the general form by elementary functions, appear in the equations for φ_k. Hence, all the principal difficulties in solving the dynamic optimum turn problem are clearly visible.

§ 6.6. PARTICULAR SOLUTION OF THE DYNAMIC OPTIMUM TURN PROBLEM FOR A BODY WITH SPHERICAL SYMMETRY AND A LIMITATION ON THE MAGNITUDE OF THE CONTROLLING TORQUE

This case is apparently uniquely resolvable analytically, since, for the spherical symmetry of the body, we have $J_1 = J_2 =$

$J_3 = J$ and the nonlinear equations (6.68) (it is necessary to take into account that the conditions for the maximum ω_k are determined as functions as φ_k) become linear differential equations. On the other hand, the solution even for such a particular case of rigid body summetry is of interest in that it permits one to trace to the end all the characteristics of the formulated problem and well illustrates the role of the solution obtained for the kinematic problem.

For the case under consideration, the Hamiltonian (6.67) / 296 takes the form

$$\mathcal{H} = \varphi_1 \frac{M_1}{J} + \varphi_2 \frac{M_2}{J} + \varphi_3 \frac{M_3}{J} + \mathcal{H}_k - |M_1| - |M_2| - |M_3| \quad (6.74)$$

(we have set $r_1 = r_2 = r_3 = 1$).

The dynamic Euler equations (5.51) and the equations for the supplementary functions φ_k (6.68) reduce to the following:

$$\dot{\omega}_k = \frac{M_k}{J}, \quad \dot{\varphi}_k = -\frac{1}{2} p_k \quad (k = 1, 2, 3). \quad (6.75)$$

One should also add here the kinematic Equations (6.1) for the quantities λ_i and the relations (6.15) determining the quantities p_k as functions of λ_i and ψ_i ($i = 0, 1, 2, 3$). We write these equations in quaternion form, by using relation (6.20):

$$\dot{\Lambda} = \frac{1}{2} \Lambda \circ \omega_E, \quad p_E = \tilde{\Lambda} \circ p_{E0} \circ \Lambda, \quad (6.76)$$

where $\mathbf{p}_{E0}$ is the initial value of the conjugate vector and $\mathbf{p}_E$ is the quaternion (6.15).

We first investigate the case of control optimization with respect to the speed of response and consider the condition for the maximum (6.72) for this case. We introduce the following vectors (and quaternions):

$$\left.\begin{aligned}\mathbf{M} &= M_1 i_1 + M_2 i_2 + M_3 i_3,\\ \boldsymbol{\varphi} &= \varphi_1 i_1 + \varphi_2 i_2 + \varphi_3 i_3.\end{aligned}\right\} \tag{6.77}$$

We note that, if the hypercomplex basis is superimposed with the basis **E**, then **M** is the controlling torque vector in the body axes. With consideration of (6.77), the conditions for the maximum (6.72) is written in the form:

$$\max \mathcal{H} = \max_{\mathbf{M}\in\mathbf{M}} [\boldsymbol{\varphi}\cdot\mathbf{M}].$$

In the particular case when the region of allowable controls is a sphere (the torque is limited in magnitude), we have from the condition for the maximum the final expression for the control function

$$\mathbf{M} = \mu\boldsymbol{\varphi}. \tag{6.78}$$

Here μ is a normalizing coefficient equaling $M_{max}/|\varphi|$, where / 297 M_{max} is the maximum value of the torque magnitude. This relation expresses the control as a function only of the supplementary variables φ: the torque **M** must be maximum in magnitude and directed along the vector $\boldsymbol{\varphi}$. The vector $\boldsymbol{\varphi}$ is determined in turn by the second equation of (6.75) as a function of the vector $\mathbf{p}_{\dot{E}}$; the vector $\mathbf{p}_{\dot{E}}$ is expressed by relation (6.76) as a function of ω_ε.

We represent the relations (6.75) in vector form, by using the designations (6.77) and considering the control determined by the condition (6.78):

$$\dot{\omega}_E = \frac{\mu}{J}\,\eta = \frac{M_{max}}{J}\,\frac{\varphi}{|\varphi|}, \quad \dot{\eta} = -\frac{1}{2}\,p_E. \tag{6.79}$$

The obtained equations, together with Equations (6.76) give a closed system of differential equations, which must uniquely satisfy the boundary conditions (6.62) and (6.63).

It should be noted that writing Equations (6.76) in vector form is a formal representation of the equalities (6.75), thus, the derivatives $\dot{\omega}_E$ and $\dot{\eta}$ are local derivatives here (taken without the consideration of the motion of the basis E).

Let us discuss the choice of the arbitrary constants for the obtained system of equations and their relation to the boundary conditions of the problem. Since the vector equalities (6.79) are a formal record of scalar relations, then the differential reqlations (6.79) can be formally integrated:

$$\left.\begin{aligned} \omega_E &= \omega_0 + \frac{M_{max}}{J}\int_0^t \frac{\varphi}{|\varphi|}\,dt', \\ \eta &= \eta_0 - \frac{1}{2}\int_0^t p_E\,dt', \end{aligned}\right| \tag{6.80}$$

where ω_0 and η_0 are the initial values of the quantities ω_E and η.

It follows from the obtained description that the first condition of (6.63) is satisfied with the choice of the initial value of the angular velocity equaling the corresponding boundary condition. One can imagine that the second condition of (6.63) can be satisfied by a corresponding choice of the initial vector / 298 η_0. The first (initial) condition of (6.62) is also expressed automatically by specification of the corresponding initial quaternion Λ_0 in the solution of the kinematic equation. The

final value of the quaternion Λ_T must satisfy the corresponding selection of the initial vector $\mathbf{p}_{E0}$ (6.76).

Thus, the obtained closed system of equations must have a solution determined by nine constants (three constants for each of the vectors ω_E, $\mathbf{q}$ and $\mathbf{p}_E$), which completely conforms with the boundary conditions of the problem.

Let us turn to an analysis of the obtained system. The vector $\mathbf{p}_E$ is obtained as a result of solving the kinematic equations. By differentiating the second relation of (6.76), we obtain the following differential equation describing the motion of this vector:

$$2\dot{p}_E = p_E \circ \omega_E - \omega_E \circ p_E. \tag{6.81}$$

It can be seen that Equations (6.79) and (6.81) exactly form a closed system of three vector differential equations of first order (i.e., a system of nine scalar equations). However, these equations are nonlinear; they contain the three variables ω_E, $\mathbf{q}$ and $\mathbf{p}_E$ and even the reduction of these equations to one variable represents serious difficulties.

With consideration of the fact that the vector equalities (6.79) are a formal description of scalar relations, we differentiate the second equation of this system:

$$\dot{\mathbf{q}} = -\frac{1}{2}\dot{p}_E.$$

We substitute the value of the derivative (6.81) into the obtained equality, replacing the function $\mathbf{p}_E$ by $\dot{\mathbf{q}}$. We will have as a result the system of equations corresponding to the system (6.79), (6.81), and containing only two variables:

$$\left.\begin{aligned}\dot{\omega}_E &= \frac{M_{max}}{J}\frac{\varphi}{|\varphi|},\\ 2\bar{\dot{q}} &= \dot{q}\circ\omega_E - \omega_E\circ\dot{q} = 2\dot{\varphi}\times\omega_E.\end{aligned}\right\} \qquad (6.82)$$

The general solution of the obtained system cannot be investigated; we will thus indicate one particular solution. Let the vector ω be parallel to the vector $\dot{\varphi}$, i.e., parallel to the vector $\mathbf{p}_E$; this means that the derivative of the vector φ has the same direction as the vector φ. Proceeding from the equality (6.81), we obtain that the following equality must occur: / 299

$$p_E = p_{E0},$$

and the vector ω_E must have a constant direction. We obtain from the second equation of (6.80):

$$\varphi(t) = -\frac{1}{2}p_{E0}t + a, \qquad (6.83)$$

where **a** is a constant vector. Since the vector ω_E must have the same direction as the vector ω_E, i.e., as the vector $\mathbf{p}_{E0}$, then the vector **a** must be collinear with the vector $\mathbf{p}_{E0}$. Thus, the function $\varphi(t)$ is linear for the case under consideration and can change sign only once. In accordance with this, the controlling torque, according to (6.78), is constant in direction, maximum in magnitude, and can change sign once during the whole trajectory of the motion.

The unit vector of the function $\varphi(t)$ coincides with the vector $\mathbf{p}_{E0}$; by using the first solution of (6,80), we obtain for the angular velocity function:

$$\omega_E = \omega_0 + \frac{M_{max}}{J}p_{E0}\left\{\begin{array}{ll}\pm t & \text{for } 0\leqslant t\leqslant\tau_1\\ \pm(2\tau_1 - t) & \text{for } \tau_1\leqslant t\leqslant T\end{array}\right\}. \qquad (6.84)$$

where T is the time interval of the motion, and τ_1 is the time at which the sign of the function $\varphi(t)$ changes. It is seen from the obtained equality that ω_0, the initial value of the angular velocity, cannot be arbitrary, but must have the direction of the vector $\mathbf{p}_{E0}$; the final value ω_T is determined by this direction in exactly the same way. Hence, the conclusion follows: the particular solution of the system (6.82) under consideration is valid only for certain boundary conditions (6.63), namely those when the initial and final velocity vectors have the direction of the vector $\mathbf{p}_{E0}$. Accordingly, the number of these / 300
conditions (6.63) decreases from six to two (i.e., only the quantities ω_0 and ω_T are specified). Nonetheless, it should be noted that such a particular solution has practical value, and the boundary conditions (6.64) are just such a case. Because of this, the number of boundary conditions (6.62) and (6.63) decreases from nine to five: three conditions (6.62) on the angle and direction of rotation and two conditions (6.63) on the magnitude of the initial and final velocities. The number of constants in the obtained solution also equal five: three constants define $\mathbf{p}_{E0}$ and $\mathbf{a}$ (i.e., the direction of the rotation vector and the time at which the sign of the velocity changes), and one constant each for the quantities ω_0 and ω_T.

The nature of the variation of $\omega_E(t)$ is presented in Figure 6.16. The velocity increase occurs with maximum acceleration, the turn time T is not fixed here, it must be determined by the mangitude of the turn angle. The solution of the kinematic equations for the case of the velocity vector constant in direction is known. We have, for $\omega_E(t)$ in the form (6.84)

$$\left.\begin{aligned}\Lambda(t)&=\Lambda_0\circ e^{\frac{1}{2}p_{E0}\left[\omega_0 t\pm\frac{M_{max}}{J}\frac{t^2}{2}\right]} \quad \text{for} \quad 0\leqslant t\leqslant\tau_1,\\ \Lambda(t)&=\Lambda_0\circ e^{\frac{1}{2}p_{E0}\left[\omega_0 t\pm\frac{M_{max}}{J}\left(2\tau_1 t-\tau_1^2-\frac{t^2}{2}\right)\right]}\\ &\qquad\qquad\qquad\qquad \text{for} \quad \tau_1\leqslant t\leqslant T.\end{aligned}\right\} \tag{6.85}$$

The total rotation angle over the time T is found as:

$$\theta = \omega_0 T \pm \frac{M_{max}}{J}\left(2\tau_1 T - \tau_1^2 - \frac{T^2}{2}\right); \qquad (6.86)$$

it must be determined by the quaternion $\tilde{\Lambda}_0 \circ \Lambda_T$ (6.46) with the help, for example, of the relation

$$2\theta = \arccos\,[\mathrm{sqal}\,(\Lambda_0 \circ \Lambda_T)].$$

The rotation direction, i.e., the vector $\mathbf{p}_{E0}$, is determined by / 301 the vector part of (6.85):

$$p_{E0} = \mathrm{vect}\,(\tilde{\Lambda}_0 \circ \Lambda_T)/\sin\frac{\theta}{2}. \qquad (6.87)$$

Obviously, the particular case under consideration describes a plane turn of the body from the initial position to the final. It is of interest to compare the obtained solution (6.84), (6.87) with the solution of (6.35) of the kinematic problem. It is seen from the comparison that just this particular case of control gives a plane turn determined by the quaternion $\tilde{\Lambda}_0 \circ \Lambda_T$

The quantities T and τ_1 are determined from the solutions (6.84) and (6.86):

$$\left.\begin{aligned} \omega_T - \omega_0 &= \mp\frac{M_{max}}{J}(T - 2\tau_1), \\ \theta - \omega_0 T &= \pm\frac{M_{max}}{J}\left[\frac{T^2}{2} - (T - \tau_1)^2\right]. \end{aligned}\right\} \qquad (6.88)$$

The conditions (6.88) lead to an algebraic quadratic equation relative to τ_1 or T, only one of the two roots of this equation has meaning. Nonetheless, the two signs of the solutions (6.86) and (6.88) correspond to different values of T and τ_1. For the upper sign, we have

$$\left.\begin{aligned} T &= 2\sqrt{\frac{J^2}{M_{max}^2}(\omega_\tau^2+\omega_0^2)+\frac{J}{M_{max}}\vartheta}-\frac{J}{M_{max}}(\omega_0+\omega_\tau),\\ \tau_1 &= \sqrt{\frac{J^2}{M_{max}^2}(\omega_\tau^2+\omega_0^2)+\frac{J}{M_{max}}\vartheta}-\frac{J}{M_{max}}\omega_0. \end{aligned}\right\} \qquad (6.89)$$

The lower sign of relations (6.88) correspond to the values:

$$\left.\begin{aligned} T &= \frac{J}{M_{max}}(\omega_0+\omega_\tau)-2\sqrt{\frac{J^2}{M_{max}^2}(\omega_0^2+\omega_\tau^2)-\frac{J}{M_{max}}\vartheta},\\ \tau_1 &= \frac{J}{M_{max}}\omega_0-\sqrt{\frac{J^2}{M_{max}^2}(\omega_\tau^2+\omega_0^2)-\frac{J}{M_{max}}\vartheta}. \end{aligned}\right\} \qquad (6.90)$$

These two solutions reflect the two paths leading from the point ω_0 to the point ω_τ (cf., Figure 6.16). Only one solution (6.89) is possible when the initial and final angular velocities / 302 are zero. Obviously, this case corresponds to the motion when the torque acts toward one side in the direction of the required turn for half the time of the motion, and in reverse for the other half of the time. In accordance with (6.88), the angle, by which the turn is accomplished with such controls, equals

$$\vartheta = \frac{M_{max}}{J}\frac{T^2}{4}.$$

and the maximum angular velocity:

$$\omega_E = \frac{M_{max}}{J}\frac{T}{2}.$$

The solution (6.90) has meaning only when the initial and final angular velocities differ from zero. It is clear that, in order that the solution (6.90) exists, the inequality

$$\omega_0^2+\omega_\tau^2 > \frac{M_{max}}{J}\vartheta.$$

must be satisfied. Then the nature of the control can be quite different: the controlling torque at the beginning of the motion can have a direction opposite to the required turn (cf., Figure 6.16).

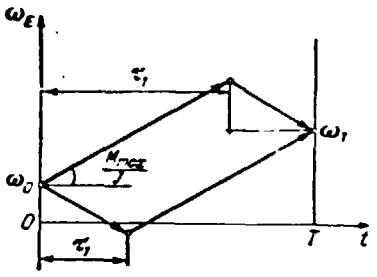

Figure 6.16.

In either case, the solution is ambiguous and expresses, firstly, the possibility of a turn by the angles ϑ or $2\pi-\vartheta$ and, secondly, that boundary conditions exist, for which both the solution (6.89) and the solution (6.90) have meaning. Obviously, one should always select from the allowable solutions that which has the shortest motion time T. The answer to this problem is not trivial, unless we have zero boundary conditions on the velocities (6.64). In this case, the solution is clearly unique: the solution (6.90) does not exist, and one should select from the angles ϑ and $2\pi-\vartheta$ the one less than π. The investigation of the general case is very interesting and can be carried out on the basis of the obtained relations; such a consideration will not be presented here, since it would lead us from the basic idea of discussion to the side of particular details.

Let us now turn to the second criterion for optimization, / 303 minimization of the controlling torque impulse for a fixed turn time T. The condition for the maximum of the Hamiltonian $\mathcal{H}$ (6.73) for this case defines a more complicated dependence of the controlling torque on the function $\varphi(T)$, than occurred for the case (6.72). It can be seen that, if

$$\mathcal{H}=\frac{1}{J}\mathbf{M}\cdot\varphi-\sum_{i=1}^{3}|M_i|>0,$$

i.e.,

$$\mathbf{M}\cdot\varphi>J\sum_{i=1}^{3}|M_i| \qquad (6.91)$$

the condition for the maximum of $\mathcal{H}$ is determined by the same relations as in the case (6.72). However, when the inequality (6.91) is violated, the maximum (non-negative) value of $\mathcal{H}$ is achieved for zero control:

$$M_i \equiv 0 \qquad (i = 1, 2, 3).$$

Thus, the motion must consist of segments, when the maximum torque acts, and segments of free motion. This can be seen that the optimum control on the segments of non-zero controlling torque will be determined by exactly the same relations as in the case of the problem of optimizing the turn time: the controlling torque is determined by the relation (6.78), the rigid body motion and the change of the function Ψ by the relations (6.79), and the kinematics of the motion are described by the equalities (6.76). Hence, it follows that the obtained solutions for the problem of optimization with respect to the speed of response is completely applicable to the separate segments of motion for the problem under consideration. The remaining segments of motion must have zero controlling torque, i.e., they are segments of free rigid body motion. The trajectories of free motion of a spherically symmetric rigid body will be:

$$\left.\begin{aligned} \omega_E(t) &= \omega_E(\tau_0), \\ \Lambda(t) &= \Lambda(\tau_0) \circ e^{\frac{1}{2} * E(\tau_0)(t-\tau_0)}, \end{aligned}\right\} \qquad (6.92)$$

where τ_0 corresponds to the time at which the free motion begins. / 304

The relations (6.84), (6.85) obtained above and the equalities (6.92) are sufficient to analyze the motion. It should be noted that the number of parameters determining the motion have increased by one (the fixed turn time is specified); this corresponds to the additional condition (6.91) giving the time of transition from one segment of motion to another. On the segments of non-zero controlling torque, we will consider,

as in the preceding case, the particular case of motion when the vector $\dot{\varphi}$ is directed along the vector φ. In this case, the vectors $\omega_\varepsilon(t)$ and $\mathbf{p}_E(t)$ maintain a constant direction. Obviously, with the transition to another segment of motion, i.e., a segment of zero torque, the subsequent motion is described by the relations (6.92), from which it follows that the angular velocity vector ω_ε maintains its position. Accordingly, with the transition to a segment of free motion, the orientation of the rotation quaternion $\Lambda(t)$ does not change and, consequently, the vector $\mathbf{p}_E(t) = \mathbf{p}_{E0}$ also remains fixed. Hence, it follows that the function $\varphi(t)$ is described by the same equation on all segments of motion and does not have discontinuities, i.e., for all segments of motion, the relation (6.83) is valid.

Since the function $\varphi(t)$ varies linearly with time, passing through zero, the inequality (6.91) can be violated only once; when the quantity $\mathbf{M}\varphi$ falls inside the cube with sides

$$M_i\left(\sum_{i=1}^{3}|M_i|\right)$$

Depending on the initial conditions, there can then be such motions as: a) motion with the maximum torque, a segment of free motion, motion with the maximum torque of opposite sign; b) motion with maximum torque, segment of free motion; c) a segment of free motion, a segment with the maximum torque.

The variation of the angular velocity ω_ε is determined on each segment of motion by the relations (6.84) and the first equality of (6.92). The nature of the variation of the angular velocity depending on the case of motion is shown in Figure 6.17. It is seen from the figure that the cases b) and c) can / 305 exist only with $\omega_0 \neq \omega_T$, with a unique trajectory for each of the cases. Case a) can be executed by two methods, which correspond to different signs of the initial control torque.

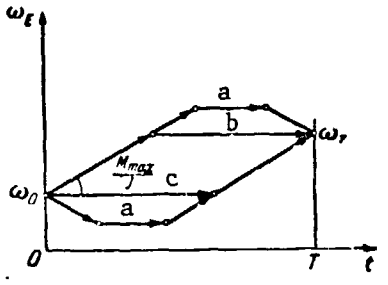

Figure 6.17.

In addition, case a) can degenerate into the motion without a free segment considered above, when the time of motion T satisfies one of the equalities (6.89) or (6.90), or into free motion only, when the following relation is satisfied:

$$\vartheta = \omega_0 T = \omega_T T. \tag{6.93}$$

In the formulation of the problem under consideration, the torque impulse and, consequently, the time of the segments of motion with maximum controlling torque are minimized. It follows from Figure 6.17 that, in all cases, the trajectories b) and c), if they are possible, are more economical than the trajectory a). Accordingly, for a given ω_0, ω_T, T and $\tilde{\Lambda}_0 \circ \Lambda_T$ [recall that the direction of the angular velocity vector in the particular case under consideration coincides with the direction of $(\tilde{\Lambda}_0 \circ \Lambda_T)$], analysis of the motion is performed in the sequence: verification of the cases b) and c) and then a selection of one of the motions a) (of which there are two, as in the preceding case).

The following relation must be satisfied for the trajectory b):

$$\omega_T - \omega_0 = \pm \frac{M_{max}}{J} \tau_1,$$

$$\omega_T (T - \tau_1) + \omega_0 \tau_1 \pm \frac{M_{max}}{J} \frac{\tau_1^2}{2} = \vartheta,$$

where τ_1 is the length of the first segment of motion with maximum torque. The value of τ_1 is found from these relations in the form:

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$$\tau_1 = \sqrt{\frac{2(\omega_\tau T - \vartheta)J}{M_{max}}}. \qquad (6.94)$$

For the rotation angle $\vartheta \geqslant \omega_\tau T$ [the condition (6.93) can occur in the case of the equality, the motion along such a trajectory is possible] and $\omega_0 \neq \omega_\tau$, rotation is accomplished by the angle $\vartheta + 2k\pi$ $(k = 1, 2, \ldots)$, unless there are more economical trajectories of the other type.

Analogous relations are fulfilled for the trajectory c):

$$\omega_\tau - \omega_0 = \pm \frac{M_{max}}{J}(T - \tau_3),$$
$$\omega_0 T \pm \frac{M_{max}}{J}\frac{(T-\tau_3)^2}{2} = \vartheta,$$

where τ_1 is the length of the first segment of free motion. The value τ_3 is found from these relations in the form:

$$\tau_3 = T - \sqrt{\frac{2(\vartheta - \omega_0 T)J}{M_{max}}}. \qquad (6.95)$$

In those cases when $\omega_0 T \geqslant \vartheta$ [the equality is possible with fulfillment of condition (6.93)], motion along such a trajectory must occur by the angle $\vartheta + 2k\pi$ $(k = 1, 2, \ldots)$, unless there are more economical trajectories of the other type. Likewise, if τ_3 is obtained negative, the motion must be accomplished by the angle $2\pi - \vartheta$ or along a trajectory of the other type.

We have, for the trajectory a), the following relation:

$$\omega_\tau - \omega_0 = \pm \frac{M_{max}}{J}(\tau_1 - \tau_2),$$
$$\pm\omega_0 T + \frac{M_{max}}{J} T\tau_1 - \frac{M_{max}}{J}\tau_1^2 - \tau_1(\omega_0 - \omega_\tau) - \frac{J}{2M_{max}}(\omega_0 - \omega_\tau)^2 = \pm\vartheta,$$

where $\tau_1 > 0$, $\tau_3 > 0$ are the lengths of the first and third segments of motion with constant maximum torque.

These relations determine the values of τ_1 and τ_2 : / 307

$$\left.\begin{aligned}
\tau_1 &= \frac{T}{2} \pm \frac{(\omega_T - \omega_0)J}{2M_{max}} - \\
&- \sqrt{\frac{T^2}{4} \pm \frac{TJ(\omega_T - \omega_0)}{2M_{max}} \mp \frac{(\vartheta - \omega_0 T)J}{M_{max}} + \frac{3J^2}{4M_{max}^2}(\omega_0 - \omega_T)^2}, \\
\tau_2 &= \frac{T}{2} \pm \frac{(\omega_0 - \omega_T)J}{2M_{max}} - \\
&- \sqrt{\frac{T^2}{4} \pm \frac{(\omega_T - \omega_0)JT}{2M_{max}} \mp \frac{(\vartheta - \omega_0 T)J}{M_{max}} + \frac{3J^2}{4M_{max}^2}(\omega_0 - \omega_T)^2}.
\end{aligned}\right\} \qquad (6.96)$$

The upper sign here corresponds to the case of motion when the first segment has a positive torque (and the third, negative); the lower sign corresponds to the reverse sequence of torques. We note that the second trajectory a) does not always occur. Thus, for example, only the first trajectory of (6.96) exists for $\omega_0 = \omega_T = 0$, since we have negative values of τ_1 and τ_2 for the second. Only the first trajectory $(\tau_2 < 0)$ also exists for $\omega_0 = \omega_T > 0$ and $\vartheta > \omega_0 T$, the trajectory of free motion only (6.93) is executed for $\vartheta = \omega_0 T$, both the first and second trajectories exist for $\vartheta < \omega_0 T$. With further increase of ω_0, the values τ_1 and τ_2 approach 1/2 T, and for $\omega_0 = \vartheta + \frac{T^2}{4}\frac{M_{max}}{J}$ become equal to 1/2 T. For $\omega_0 > \vartheta + \frac{T^2}{4}\frac{M_{max}}{J}$ the solutions (6.96) formally lose meaning (a negative quantity appears under the root). This indicates that a turn in the time T by the angle ϑ is impossible for such boundary angular velocities; the turn is then accomplished by the angle $\vartheta + 2k\pi$ $(k = 1, 2, \ldots)$.

In either case, there exist several trajectories of type a), which accomplish the required turn. Such ambiguity is determined, firstly, by the possibility of a turn by the angles ϑ, $2\pi - \vartheta$ and $\vartheta + 2k\pi$ $(k = 1, 2, \ldots)$, which give the same spatial turn and, secondly,

by the two possible trajectories (6.96). All possible trajectories should always be examined to find the optimum turn and the most economical selected from them. / 308

Such ambiguity exists not only for zero boundary conditions on the velocities (6.64). In this case, as was indicated above, only one trajectory (6.96) is executed, the equality $\tau_1 = \tau_2$ occurs and the optimum turn is accomplished by the angles $\theta < \pi$. Such a case has practical value, thus we will consider it in more detail. For such motion, the acceleration occurs from the beginning with maximum controlling torque, which has the direction of the mismatch quaternion $\tilde{\Lambda}_0 \circ \Lambda_T$, up to the value of angular velocity:

$$\omega = \frac{TM_{max}}{2J}\left(1 - \sqrt{1 - \frac{4\theta J}{T^2 M_{max}}}\right), \qquad (6.97)$$

which is determined from the given turn time. A segment of three motion of length

$$T\sqrt{1 - \frac{4\theta J}{T^2 M_{max}}}$$

follows after this, and then a segment of braking with the maximum torque in the opposite direction. In practice, the angular velocity of the turn (6.97), and not the turn time T, is usually specified. If the controlling torque is then specified in the form of the relay function:

$$M_i = M_i(\omega_{Ei} - \Omega_{ki}) \quad (i = 1, 2, 3), \qquad (6.98)$$

where Ω_{ki} is the correction angular velocity (6.37), then it is seen that the control (6.98) executes exactly the optimum control considered here for the case of the spherically symmetric body. This explains the assertion made in Chapter 5 that the control by the quaternion components is close to optimum.

§ 6.7. SUPPLEMENTARY REMARKS

I. We will briefly discuss the hypotheses made in paragraph 6.4. The hypothesis z is the simplest of them. The investigation carried out in this paragraph has shown that, for each specified Λ_z, there can exist one or several types of trajectories, which execute this motion. Accordingly, Λ_z determines / 309 one or several values of $\mathbf{p}_{E0}$. Actually, in any finite region, one trajectory will be optimum (with consideration of the sufficient conditions for optimization), thus Λ_z actually determines the vector $\mathbf{p}_{E0}$ uniquely. However, we have not been successful in finding an algorithm, which sets a vector $\mathbf{p}_{E0}$ in correspondence with the quaternion Λ_z.

The hypothesis 3 is more complicated; its alternative is the assumption that the vector $\mathbf{p}_E(t)$ can change its sign in the motion process (i.e., a transition can occur from one of the two values to the other). If such a possibility is assumed, then it can be shown that such a transition from one sign to another can occur only when the vector $\mathbf{p}_E$ falls on the coordinate plane.

In this case, motion is possible in the sliding mode when the vector $\mathbf{p}_E$ moves along the coordinate plane (Figure 6.18); the angular velocity of such motion will equal $\omega_{\Lambda n}$. The length of such a segment depends on the place at which the vector $\mathbf{p}_E$ falls on the coordinate plane and does not exceed $\frac{\pi}{4}$, since such a sliding mode continues up to the incidence of the vector $\mathbf{p}$ at the point $\mathscr{C}_1$. The motion with the velocity $\omega_{\Lambda n}$ in this case can be joined with other motions; the number of possible trajectories increases because of the appearance of new types of trajectories included as one of the segments of motion (6.50). The following thirteen trajectories are such motions (we use here the notation for the component quaternions (6.48) — (6.50)): / 310

$$\Lambda_{\varphi 1}\circ\Lambda_1,\ \Lambda_{\varphi 1}\circ\tilde{\Lambda}_2,\ \Lambda_1\circ\Lambda_{\varphi 1},\ \Lambda_1\circ\Lambda_{\varphi 1}\circ\Lambda_1',\ \Lambda_1\circ\Lambda_{\varphi 1}\circ\tilde{\Lambda}_2,$$
$$\Lambda_4\circ\Lambda_1\circ\Lambda_{\varphi 1},\ \Lambda_4\circ\Lambda_1\circ\Lambda_{\varphi 1}\circ\Lambda_1',\ \Lambda_4\circ\Lambda_1\circ\Lambda_{\varphi 1}\circ\tilde{\Lambda}_2,$$
$$\Lambda_3\circ\Lambda_4\circ\Lambda_1\circ\Lambda_{\varphi 1},\ \Lambda_3\circ\Lambda_4\circ\Lambda_1\circ\Lambda_{\varphi 1}\circ\Lambda_1',$$
$$\Lambda_3\circ\Lambda_4\circ\Lambda_1\circ\Lambda_{\varphi 1}\circ\tilde{\Lambda}_2,\ \Lambda_{\varphi 1}\circ\Lambda_1\circ\Lambda_5,\ \Lambda_{\varphi 1}\circ\tilde{\Lambda}_2\circ\Lambda_6.$$

All these types of trajectories permit one to determine uniquely the component rotations by the resultant quaternion Λ_Σ and the conditions of the motion of the vector $\mathbf{p}_E$. Thus, for example, on the trajectory $\Lambda_4\circ\Lambda_1\circ\Lambda_{\varphi 1}\circ\Lambda_1'$, the number of segments of which exceeds three, the angle $\theta_{\varphi 1}$ and θ_1 are related by the condition:

$$\sqrt{3}\,\mathrm{tg}\left(\frac{\pi}{4}-\theta_{\varphi 1}\right)=\mathrm{ctg}\left(\frac{\theta_1}{2}+\frac{\pi}{6}\right)$$

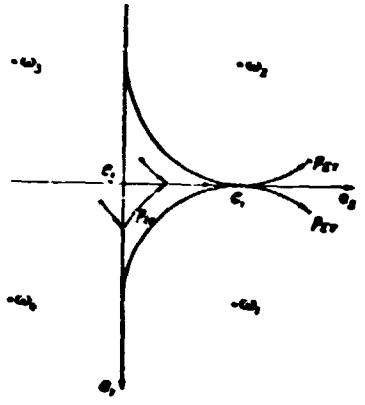

Figure 6.18.

(The meaning of this condition is taken from Figure 6.18). Besides this, there exist another whole series of trajectories, in which both the rotation $\Lambda_{\varphi n}$ and the rotation $\Lambda_{\gamma 1}$ are present among the component rotations. For such trajectories, the component rotations cannot be determined in all cases. We will indicate the trajectories, for which Λ_Σ uniquely determines all the component rotations

$$\Lambda_{\varphi 1}\circ\Lambda_{\gamma 1},\ \Lambda_1\circ\Lambda_{\varphi 1}\circ\Lambda_{\gamma 1},\ \Lambda_4\circ\Lambda_1\circ\Lambda_{\varphi 1}\circ\Lambda_{\gamma 1},$$
$$\Lambda_3\circ\Lambda_4\circ\Lambda_1\circ\Lambda_{\varphi 1}\circ\Lambda_{\gamma 1}.$$

A numerical investigation was carried out in all the cases of motion cited here. Regions of existence for the solutions for each type of trajectory were constructed; these regions are superimposed on the regions existing for the motion of the vector

$\mathbf{p}_E$ in the region $\mathcal{N}_n$ and along the separatrix and those considered in paragraph 6.4. The results of the calculation have shown that all these trajectories are "longer" than those considered in paragraph 6.4. Sometimes it appeared that the trajectories of such a nature have longer motion times than for the control (6.37). Thus, the validity of hypothsis 3 is verified by the performed investigation.

The hypothesis 1 is, evidently, the most complicated and unobvious. If it is assumed that the function sign (0) in the expression (6.44) does not have a zero among its values, then / 311
motions with the velocities $\omega_{\kappa l}$ are excluded. Then the trajectories of the type XIII — XIV vanish and a number of regions appear which are not covered by the remaining trajectories. Thus, in order to have optimum control at any point (the point of the positions of the ends of the vector Λ_Σ), it is necessary to assume a zero value among the values of the function sign (0). Analogously, the absence of the values +1 or -1 for this same function sign (0) leads to a reduction of the possible motions, as a result of which regions appear in which there are no "optimum" solutions. Nonetheless, it is difficult to understand physically, with what form the shift of the values of the function sign (0) occurs with motion of the vector $\mathbf{p}_E$.

The hypothesis 3 permits one to examine only one hemisphere, in which the ends of the vectors Λ_Σ are arranged. The quaternion Λ_Σ, having the opposite direction, corresponds to the inverse rotation.

It can be seen that the motion executed by the second value of the vector $\mathbf{p}_E$ must correspond to the inverse rotation $\tilde{\Lambda}_\Sigma$, i.e., the rotation will occur with the inverse (in sign) angular velocities. However, this same motion, if it is examined on the second hemisphere, will appear exactly the same as is stipulated by the hypothesis 3.

II. We will show that the trajectory of type XII for the middle rotation angle equaling $\frac{2\pi}{3}$ becomes equivalent to the trajectories of type VIII and IX. We will consider first the quaternion of the trajectory of type IX:

$$\Lambda_\Sigma = \Lambda_{\gamma 1} \circ \Lambda_2,$$

and substitute here the values of the components, in accordance with (6.54) and (6.56).

By performing the multiplication, we obtain for the components of the resultant quaternion:

$$\lambda_{\Sigma 0} = \cos\frac{\theta_{\gamma 1}}{2}\cos\frac{\theta_2}{2} - \sqrt{\frac{2}{3}}\sin\frac{\theta_{\gamma 1}}{2}\sin\frac{\theta_2}{2},$$
$$\lambda_{\Sigma 1} = -\frac{1}{\sqrt{3}}\cos\frac{\theta_{\gamma 1}}{2}\sin\frac{\vartheta_2}{2}.$$

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$$\lambda_{\Sigma 2} = \frac{1}{\sqrt{2}}\sin\frac{\theta_{\gamma 1}}{2}\cos\frac{\theta_2}{2} + \frac{1}{\sqrt{3}}\cos\frac{\theta_{\gamma 1}}{2}\sin\frac{\theta_2}{2} - \frac{1}{\sqrt{6}}\sin\frac{\theta_{\gamma 1}}{2}\sin\frac{\theta_2}{2},$$
$$\lambda_{\Sigma 3} = \frac{1}{\sqrt{2}}\sin\frac{\theta_{\gamma 1}}{2}\cos\frac{\theta_2}{2} + \frac{1}{\sqrt{3}}\cos\frac{\theta_{\gamma 1}}{2}\sin\frac{\theta_2}{2} + \frac{1}{\sqrt{6}}\sin\frac{\theta_{\gamma 1}}{2}\sin\frac{\theta_2}{2}.$$

We now multiply the obtained quaternion on the right by the quaternion $\Lambda_{\gamma 2}$ [in accordance with (6.56)]. After the necessary transformations, it reduces to the form:

$$\Lambda_\Sigma = \Lambda_{\gamma 1} \circ \Lambda_2 \circ \Lambda_{\gamma 2},$$
$$\lambda_{\Sigma 0} = \cos\frac{\theta_2}{2}\cos\frac{\theta_{\gamma 1} + \theta_{\gamma 2}}{2} - \sqrt{\frac{2}{3}}\sin\frac{\theta_2}{2}\sin\frac{\theta_{\gamma 1} + \theta_{\gamma 2}}{2} + \frac{1}{\sqrt{3}}\sin\frac{\theta_{\gamma 1}}{2}\sin\frac{\theta_{\gamma 2}}{2}\sin\left(\frac{\pi}{3} - \frac{\theta_2}{2}\right),$$
$$\lambda_{\Sigma 1} = -\frac{1}{\sqrt{3}}\sin\frac{\theta_2}{2}\cos\frac{\theta_{\gamma 1} + \theta_{\gamma 2}}{2} +$$

$$+\sin\left(\frac{\vartheta_2}{2}-\frac{\pi}{3}\right)\left[\frac{1}{\sqrt{3}}\sin\frac{\vartheta_{\gamma 2}}{2}\left(\sqrt{2}\cos\frac{\vartheta_{\gamma 1}}{2}-\sin\frac{\vartheta_{\gamma 1}}{2}\right)\right].$$

$$\lambda_{\Sigma 2}=\frac{1}{\sqrt{2}}\cos\frac{\vartheta_2}{2}\sin\frac{\vartheta_{\gamma 1}+\vartheta_{\gamma 2}}{2}+$$
$$+\frac{1}{\sqrt{3}}\sin\frac{\vartheta_2}{2}\cos\frac{\vartheta_{\gamma 1}+\vartheta_{\gamma 2}}{2}-\frac{1}{\sqrt{6}}\sin\frac{\vartheta_2}{2}\sin\frac{\vartheta_{\gamma 1}+\vartheta_{\gamma 2}}{2}+$$
$$+\frac{1}{\sqrt{3}}\sin\left(\frac{\vartheta_2}{2}-\frac{\pi}{3}\right)\sin\frac{\vartheta_{\gamma 2}}{2}\left(\sqrt{2}\cos\frac{\vartheta_{\gamma 1}}{2}+\sin\frac{\vartheta_{\gamma 1}}{2}\right).$$

$$\lambda_{\Sigma 3}=\frac{1}{\sqrt{2}}\cos\frac{\vartheta_2}{2}\sin\frac{\vartheta_{\gamma 1}+\vartheta_{\gamma 2}}{2}+$$
$$+\frac{1}{\sqrt{3}}\sin\frac{\vartheta_2}{2}\cos\frac{\vartheta_{\gamma 1}+\vartheta_{\gamma 2}}{2}+\frac{1}{\sqrt{6}}\sin\frac{\vartheta_2}{2}\sin\frac{\vartheta_{\gamma 1}+\vartheta_{\gamma 2}}{2}+$$
$$+\frac{1}{\sqrt{3}}\sin\frac{\vartheta_{\gamma 1}}{2}\sin\frac{\vartheta_{\gamma 2}}{2}\sin\left(\frac{\pi}{3}-\frac{\vartheta_2}{2}\right).$$

If one sets $\vartheta_{\gamma 1}=0$ here, then we obtain the resultant quaternion for the trajectory of type VIII: / 313

$$\Lambda_\Sigma=\Lambda_2\circ\Lambda_{\gamma 2},$$

$$\lambda_{\Sigma 0}=\cos\frac{\vartheta_2}{2}\cos\frac{\vartheta_{\gamma 2}}{2}-\sqrt{\frac{2}{3}}\sin\frac{\vartheta_2}{2}\sin\frac{\vartheta_{\gamma 2}}{2}.$$

$$\lambda_{\Sigma 1}=-\frac{1}{\sqrt{3}}\sin\frac{\vartheta_2}{2}\cos\frac{\vartheta_{\gamma 2}}{2}-\frac{1}{\sqrt{2}}\cos\frac{\vartheta_2}{2}\sin\frac{\vartheta_{\gamma 2}}{2}+$$
$$+\frac{1}{\sqrt{6}}\sin\frac{\vartheta_2}{2}\sin\frac{\vartheta_{\gamma 2}}{2}.$$

$$\lambda_{\Sigma 2}=\frac{1}{\sqrt{3}}\sin\frac{\vartheta_2}{2}\cos\frac{\vartheta_{\gamma 2}}{2}.$$

$$\lambda_{\Sigma 3}=\frac{1}{\sqrt{2}}\cos\frac{\vartheta_2}{2}\sin\frac{\vartheta_{\gamma 2}}{2}+\frac{1}{\sqrt{3}}\sin\frac{\vartheta_2}{2}\cos\frac{\vartheta_{\gamma 2}}{2}+$$
$$+\frac{1}{\sqrt{6}}\sin\frac{\vartheta_2}{2}\sin\frac{\vartheta_{\gamma 2}}{2}.$$

It can be seen that, if one sets $\vartheta_2=\frac{2\pi}{3}$ in the expression for $\Lambda_\Sigma=\Lambda_{\gamma 1}\circ\Lambda_2\circ\Lambda_{\gamma 2}$, then it exactly coincides with the expression for $\Lambda_\Sigma=\Lambda_{\gamma 1}\circ\Lambda_2$, in which the angle $\vartheta_{\gamma 1}$ should be replaced by the angle $\nu_{\gamma 1}+\vartheta_{\gamma 2}$. One can verify that the trajectory $\Lambda_\Sigma=\Lambda_2\circ\Lambda_{\gamma 2}$ also changes into the trajectory $\Lambda_\Sigma=\Lambda_{\gamma 1}\circ\Lambda_2\circ\Lambda_{\gamma 2}$ for this same value of the angle ϑ_2.

Thus, it has been shown that the quality

$$\Lambda_{\varepsilon 1}(\vartheta_{\varepsilon 1}) \circ \Lambda_2(\vartheta_2) \circ \Lambda_{\varepsilon 2}(\vartheta_{\varepsilon 2}) = \Lambda_{\varepsilon 1}(\vartheta_{\varepsilon 1} + \vartheta_{\varepsilon 2}) \circ \Lambda_2(\vartheta_2) = \\ = \Lambda_2(\vartheta_2) \circ \Lambda_{\varepsilon 2}(\vartheta_{\varepsilon 1} + \vartheta_{\varepsilon 2}).$$

is satisfied for $\vartheta_2 = \frac{2\pi}{3}$ (expletive deleted).

BASIC NOTATION

$\mathbf{H}$ — hypercomplex space;

$\mathbf{I}$, $\mathbf{E}$, $\mathbf{E}'$, $\mathbf{E}''$ — orthonormal bases in three-dimensional space;

$\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ — hypercomplex units; basis vectors of the basis $\mathbf{I}$;

$\mathbf{e}_i$, $\mathbf{e}_i'$, $\mathbf{e}_i''$ — basis vectors of bases;

$\mathbf{\Lambda}$, $\mathbf{M}$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{P}$, $\mathbf{N}$ — quaternions;

λ_i, μ_i, r_i, s_i, ρ_i, ν_i — quaternion components;

sqal, vect — designation of the scalar and vector parts of a quaternion;

λ, μ, r, s, ρ, ν — vectors; vector parts of a quaternion;

$\tilde{\Lambda}$, $\tilde{\mathbf{M}}$ — conjugate quaternions;

$\mathbf{\Lambda}^{-1}$, $\mathbf{M}^{-1}$ — inverse quaternions;

$\|\mathbf{\Lambda}\|$, $\|\mathbf{M}\|$ — norms of quaternions;

$|\mathbf{\Lambda}| = \sqrt{\|\mathbf{\Lambda}\|} = \Lambda$ — modulus (tensor) of a quaternion;

ϑ, ζ — angle and unit vector of a quaternion;

$\Delta\mathbf{\Lambda}$, $\Delta\mathbf{N}$ — infinitesimal transformation quaternions;

$\mathbf{\Lambda}_E$, $\mathbf{R}_E$ — quaternions mapping onto a basis;

$\mathbf{\Lambda}_0$, $\mathbf{\Lambda}_\tau$ — initial and final values of quaternions;

$\mathbf{\Lambda}_\Sigma$ — resultant quaternion;

ϑ_Σ, ζ_Σ — angle and unit vector of the resultant quaternion;

$\theta = 2\zeta \operatorname{tg}\frac{\vartheta}{2}$ — finite rotation vector;

$\theta = \vartheta\zeta$ — true rotation vector;

θ^* — apparent rotation vector;

$\mathbf{C}_\mathbf{I}$, $\mathbf{C}_\mathbf{E}$ — constant quaternions;

γ_1, γ_2, γ_3 — direction cosines of the vectors ζ, ζ_Σ ;

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{r}$, $\mathbf{m}$ — vectors;

$a = |a|$ — magnitude of a vector;

ω — angular velocity vector;

$\mathbf{r}_E$, $\mathbf{a}_E$, ζ_E — vectors mapping onto the basis **E**;

r_{iE}, ω_{iE} — components of the mapping vectors $\mathbf{r}_E$, ω_E;

ω_i, Ω_i — components — projections of ω onto the body and fixed bases **E** and **I**;

ω_E, ω_I — mappings of ω onto the bases **E** and **I**;

e — unit vector;

$\omega_m, \omega_{\zeta l}, \omega_{An}$ — extremum angular velocity vectors;

$\Lambda_m, \Lambda_{\zeta l}, \Lambda_{An}$ — extremum rotational velocity quaternions;

N(t) — quaternion — particular solution of the kinematic equation, including the unit initial conditions; / 315

A, **B**, **C**, **D**, **G**, $^{\varphi}$ — orthogonal matrices;

$\mathbf{A}^T$ — transposed matrix;

$\mathbf{A}^{-1}$ — inverse matrix;

R, **T**, **I**, **E** — column matrices;

$1 = \|\delta_{ij}\|$ — unit matrix (including the two-dimensional);

$1 + \epsilon$ — infinitesimal transformation matrix;

ϑ — rotation angle of the matrix **A**;

λ_k — eigenvalues of the matrix **A**;

U, **R** — two-dimensional unitary matrices;

u, v — elements of two-dimensional complex space;

U* — the complex conjugate matrix of the matrix **U**;

$\mathbf{U}^+$, $\mathbf{U}^{*T}$ — the Hermitian conjugate matrix of the matrix **U**;

$\Delta\mathbf{U}$ — infinitesimal transformation matrix;

p_1, p_2, p_3 — Pauli spin matrices;

$\Omega, \Omega_E, \Omega_I$ — three-dimensional angular velocity matrices;

V, $\mathbf{V}_E$, $\mathbf{V}_I$ — two-dimensional unitary angular velocity matrices;

π, σ, τ — versors (unit vectors);

ϑ, φ, ψ — Euler angles;

$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{I}, \ldots$ — points on a sphere;

h, τ — integration increment and time within the increment;

$\Phi = \int_0^t \omega\, dt$ — total rotation angle;

g — increment of the total rotation angle;

$\chi = \frac{g_{av}}{\varepsilon}$ — coefficient;

W, T, V — Lyapunov functions;

Ω_{ki} — correction velocity;

$\mathcal{H}$ — Hamiltonian;

ψ_i — auxiliary functions corresponding to λ_i;

φ_i — auxiliary functions corresponding to ω_i;

J_1, J_2, J_3 — principal moments of inertia of the body;

$\mathbf{M}$ — external controlling torque;

ξ_i — response axes of the angular position sensors;

$\mathbf{M}$, Ξ_E, Ξ_I — permissible control regions in dynamic and kinematic problems;

$\mathcal{A}_n$, $\mathcal{B}_m$ — regions on a sphere;

$\circ$, $\cdot$, $\times$ — signs of quaternion, scalar and vector multiplication, respectively;

ε — quantum of angular velocity measurement;

i — imaginary unit.

REFERENCES

1. Barbashin, Ye. A., and V. A. Tabuyeva. Dinamicheskiye sistemy s tsilindricheskim fazovym prostranstvom (Dynamic Systems with Cylindrical Phase Space). Moscow, "Nauka" Press, 1969.

2. Bezhko, A. P., V. N. Branets, Yu. M. Zakharov, and I. P. Shmyglevskiy. Application of Quaternions to the Theory of Finite Rigid Body Rotation. Izvestiya, AN SSSR, Mekhanika tverdovo tela, No. 1, 1971.

3. Berezin, I. S., and N. B. Zhidkov. Metody vychisleniy, I i II (Calculation Methods, I and II). Moscow, State Physics and Mathematics Press, 1959.

4. Bogolyubov, N. N, and Yu. A. Mitropol'skiy. Asimptoticheskiye metody v teorii nelineynykh kolebaniy (Asymptotic Methods in the Theory of Nonlinear Oscillations). Moscow, State Phsyics and Mathematics Press, 1958.

5. Bodanskiy, Ye. D., and V. D. Furman. On the Errors in the Numerical Integration of the Poisson Kinematic Equations. Kosmicheskiye issledovaniya, Vol .VIII, No. 6, 1970.

6. Boltyanskiy, V. G. Matematicheskiye metody optimal'novo upravleniya (Mathematical Methods of Optimum Control). Moscow, "Nauka" Press, 1966.

7. Branets, V. N., and I. P. Shmyglevskiy. Kinematic Rotation Problems in a Rotating Coordinate System. Izv. AN SSSR, Mekhanika tverdovo tela, No. 6, 1972.

8. Branets, V. N., and I. P. Shmyglevskiy. Application of Quaternions to the Control of the Angular Position of a Rigid Body. Izv. AN SSSR, Mekhanika tverdovo tela, No. 4, 1972.

9. Bukhgol'ts, N. N. Osnovnoy kurs teoreticheskoy mekhaniki, ch. II (Basic Course in Theoretical Mechanics, Part II). Moscow, State Technical and Theoretical Press, 1945.

10. Voronov, A. A., et al. Tsifrovyye analogi dlya sistem avtomaticheskovo upravleniya (Digital Analogs for Automatic Control Systems). Moscow, AN SSSR Press, 1960.

11. Vychislitel'naya tekhnika (Calculation Techniques). Handbook edited by G. D. Khaski and G. A. Korn, I and II, translated from English. Moscow, Leningrad, "Energiya" Press, 1964.

12. Hankel, H. Teoriya komplekcnykh chislovykh sistem, preimyshchestvenno obyknovennykh mnimykh chisel i kvaternionov Gamil'tona vmeste s ikh goemetricheskim tolkovaniyem (Theory of Complex Numerical Systems, Principally the Oridinary Imaginary Numbers and Hamilton's Quaternions Together with their Geometric Interpretation), translated from German, Kazan, Kazan University Press, 1912.

13. Garbuzov, A. R. Errors in Transforming Coordinates on the Plane by an Axis Rotation with a Digital Differential Analyzer. In the collection: Avtomatika i teleinformatsionnyye systemy (Automation and Remote Information Systems). Moscow, "Nauka" Press, 1965.

14. Gel'fand, I. M., R. A. Minlos, and Z. Ya. Shapiro. Predstavleniya gruppy vrashcheniy i gruppy Lorentsa (Representations of the Rotation Group and the Lorentz Group). Moscow, State Physics and Mathematics Press, 1958.

15. Gilmor, J. P., and J. Feldman. Gyroscope with Torque Feedback for a Cardanless System. Translated from English. In the collection of translations:Vorprosy raketnoy tekhniki (Problems of Rocket Technology), No. 9, 1970.

16. Goldstein, H. Klassicheskaya mekhanika (Classical Mechanics). Translated from English. Moscow, State Technical and Theoretical Press, 1957.

17. Grobov, V. A., and A. V. Kotsyuba. On the Application of the Cayley-Klein Parameters for the Study of Quasi-Precession Motion of a Free Rigid Body. Prikladnaya mekhanika, Vol. VII, No. 7, 1971. / 317

18. Gurman, V. I., E. K. Lavrovskiy, and S. I. Sergeyev. Optimum Orientation Control of an Axially-Symmetric Rotating Spacecraft. Kosmicheskiye issledovaniya, Vol. VIII, No. 8, 1970.

19. Dimentberg, F. M. Vintovoye ischisleniye i yevo prilozheniya v mekhanike (Helical Calculus and Its Applications in Mechanics). Moscow, "Nauka" Press, 1965.

20. Dimentberg, F. M., and Ye. G. Sevost'yanova. To the Theory of Finite Rigid Body Motions. Izv. AN SSSR, Mekhanika tverdovo tela, No. 2, 1969.

21. Sommerfeld, A. Mekhanika (Mechanics). Translated from German. Moscow, Foreign Literature Press, 1947.

22. Intertial Systems without Gyrostabilized Platforms (Review). Translated from English. In the collection of translations: Voprosy raketnoy tekhniki (Problems of Rocket Technology). No. 1, 1967.

23. Ishlinskiy, A. Yu. Mekhanika giroskopicheskikh sistem (Mechanics of Gyroscopic Systems). Moscow, AN SSSR Press, 1963.

24. Kalyayev, A. V. Teoriya tsifrovykh integriruyushchikh mashin i struktur (Theory of Digital Integrating Machines and Structures). Moscow, "Sovetskoye radio" Press, 1970.

25. Kalyayev, A. V. Numerical Methods for Stieltjes Integration in Digital Integrating Machines. Kibernetiki, No. 2, 1966.

26. Klein, F. Elementarnaya matematika s tochki zreniya vysshey, I i II (Elementary Higher Mathematics, I and II). Translated from German. Moscow, Leningrad, State Technical and Theoretical Press, 1933-1934.

27. Koshlyakov, V. N. On the Application of the Rodriguez-Hamilton and the Cayley-Klein Parameters to the Applied Theory of Gyroscopes. Prikladnaya matematika i mekhanika, Vol. 29, No. 4, 1965.

28. Lagalli, M. Vektornoye ischisleniye (Vector Calculus). Translated from German. Moscow, United Scientific and Technical Presses, 1936.

29. Levi-Civita, T., and U. Amaldi. Kurs teoreticheskoy mekhaniki (Course in Theoretical Mechanics), Vol. II, Chapter 2, Translated from Italian. Moscow, Foreign Literature Press, 1951.

30. Letov, A. M. Ustoychivost' nelineynykh reguliruyemykh sistem (Stability of Nonlinear Controlled Systems). Moscow, State Physics and Mathematics Press, 1962.

31. Lur'ye, A. I. Analiticheskaya mekhanika (Analytical Mechanics). Moscow, State Physics and Mathematics Press, 1961.

32. Mayorov, S. A., and G. I. Novikov. Malogabaritnyye vychislitel'nyye mashiny (Small-Scale Computers). Moscow, "Mashinostroyeniye" Press, 1967.

33. Mayorov, F. V. Elektronnyye tsifrovyye integriruyushchiye mashiny (Digital Integrating Machines). Moscow, State Scientific and Technical Machinery Press, 1962.

34. Malkin. I. G. Nekotoryye zadachi teorii nelineynykh kolebaniy (Some Problems of the Theory of Nonlinear Oscillations). Moscow, State Technical and Theoretical Press, 1956.

35. Matthews, F. B., and G. R. Taylor. Tests of a Platformless Inertial Guidance System. Translated from English. In the collection of translations:Voprosy raketnoy tekhniki (Problems of Rocket Technology), No. 8, 1970.

36. Neslukhovskiy, K. S. Tsifrobyye differentsial'nyye analizatory (Digital Differential Analyzers). Moscow, State Physics and Mathematics Press, 1963.

37. Nikolayev, I. A. On Preserving Lyapunov Stability when Simulating a Linear System of Differential Equations on a Digital Integrating Machine. Izv. vuzov (Izvestiya vyschikh uchebnykh zavedeniy), Priborostroyeniye, No. 8, 1968.

38. Otten. Inertial Guidance Systems without Gyrostabilized Platforms. Translated from English. In the collection of translations:Voprosy raketnoy tekhniki (Problems of Rocket Technology), No. 12, 1967.

39. Pars, L. A. Analiticheskaya dinamika (Analytical Dynamics). Translated from English. Moscow, "Nauka" Press, 1971. / 318

40. Pontryagin, L. S. Obyknovennyye differentsial'nyye uravneniya (Ordinary Differential Equations). Moscow, State Physics and Mathematics Press, 1961.

41. Pontryagin, L. S., V. G. Boltyanskiy, R. V. Gamkrelidze, and Ye. F. Mishchenko. Matematicheskaya teoriya optimal'nykh protsessov (Mathematical Theory of Optimum Processes). Moscow, State Physics and Mathematics Press, 1961.

42. Raushenbakh, B. V. On Measuring the Orientation Angles of Inertial Sensors. Kosmicheskiye issledovaniya, Vol. IX, No. 5, 1971.

43. Savage, P. G. A New Solution of the Second Approximation for the Orientation Calculator of a Cardanless Inertial System. Translated from English. In the collection of translations:Mekhanika (Mechanics), No. 1, 1968.

44. Singh, J. L. Klassicheskaya dinamika (Classical Dynamics). Translated from English. Moscow, State Physics and Mathematics Press, 1963.

45. Tokar', Ye. N. Possible Principles of Spacecraft Orientation Relative to a Rotating Coordinate System. Kosmicheskiye issledovaniya, Vol. IV, No. 3, 1966.

46. Tsifrobuue differentsial'nyye analizatory (Digital Differential Analyzers). Collection of translations. Moscow, Foreign Literature Press, 1959.

47. Shileyko, A. V. Tsifrobyye modeli (Digital Models). Moscow, "Energiya" Press, 1964.

48. Bortz, J. E. A New Mathematical Formulation for Strap-Down Inertial Navigation. IEEE Trans. Aerosp. and Electr. Systems, No. 1, 1971.

49. Brand, L. Vector and Tensor Analysis. New York, J. Wiley; London, Chapman and Hall, 1948.

50. Childs, D. W., B. D. Tapley, and W. T. Fowler. Suboptimal Attitude Control of a Spin-Stabilized Axisymmetric Spacecraft. IEEE TRans. Automat. Contr., Vol. 14, No. 6, 1969.

51. Crubin, C. Derivition of the Quaternion Scheme Via the Euler Axis and Angle. Internat. J. Control., Vol. 8, No. 3, 1968.

52. Dickinson. Comparison of Digital Differential Analyzer and General Purpose Equipment in Guidance Systems. Communications and Electrons, Vol. 1, No. 52, 1961.

53. Garrad, W. L. A Method for Suboptimal Stabilization of Spacecraft Angular Velocity. Internat. J. Control., Vol. 8, No. 3, 1968.

54. Hamilton, W. R. Elements of Quaternions. New York, Chelsea Publishing Company, 1969.

55. Hamilton, W. R. Lectures on Quaternions. Dublin, Hodges and Smith, 1853.

56. Ickes, B. P. A New Method of Performing Digital Control System Attitude Computation Using Quaternions. AIAA Journal, Vol. 8, No. 1, 1970.

57. Mason, W. L., and D. E. Wedekind. Prediction and Measurement of Strap-Down Inertial Measurement Unit Performance on Lunar Mission. AIAA Paper No. 1028, 1970.

58. McKern, R., and J. Gilmore. A Redundant Strap-Down Inertial System Mechanization. SIRU AIAA Paper No. 1027, 1970.

59. Mitchell, E. L., and A. E. Rogers. Quaternion Parameters in the Simulation of a Spinning Rigid Body. Simulation, New York, 1968.

60. Mortensen, R. A Globally Stable Linear Attitude Regulator./ 319 Internat. J. Control, Vol. 8, No. 3, 1968.

61. Porcelli, G. Large Angle Attitude Control of a Spinning Space Body. IEEE Trans Automat. Contr., Vol. 14, No. 6, 1969.

62. Quasins, G. R. Strap-Down Inertial Systems. Space/ Aeronautics, Vol. 40, No. 3, 1963.

63. Robinson, A. C. On the Use of Quaternions in Simulation of Rigid Body Motion. WADD Techn. Report No. 58-17, 1957.

64. Smyth, R. K. Guidance and Control 1968. AIAA Paper No. 72, 1969.

65. Weber, H. J. Inertial Guidance System Uses Digital Integrator. Space/Aeronautics, Vol. 30, No. 5, 1958.

66. Whittaker, E. T. Analytische Dynamic der Punkte and Starren Koerper (Analytical Dynamics of Particles and Rigid Bodies). Berlin, Springer, 1924.

67. Wilcox, J. C. A New Algorithm for Strapped-Down Inertial Navigation. IEEE Trans. Aerosp. and Electr. Systems, Vol. 3, No. 5, 1967.

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